ON CONFORMALLY INVARIANT SYSTEMS OF SECOND ORDER DIFFERENTIAL OPERATORS ASSOCIATED TO MAXIMAL PARABOLICS OF QUASI-HEISENBERG TYPE

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Abstract. In this paper we close the cases that were left open in our earlier works on the study of conformally invariant systems of second order differential operators. More precisely, for these cases, we find the special values of the conformally invariant systems and determine standardness of homomorphisms between generalized Verma modules coming from the systems of operators.

1. Introduction

Conformally invariant systems are systems of differential operators that are equivariant under an action of a Lie algebra. More precisely, let $\mathcal{V} \to M$ be a vector bundle over a smooth manifold M and $\mathfrak{g}_0$ a Lie algebra of first order differential operators acting on smooth sections on $\mathcal{V}$. A linearly independent list $D_1, \ldots, D_m$ of differential operators on $\mathcal{V}$ is then said to be *conformally invariant* if for all $X \in \mathfrak{g}_0$ there exist smooth functions C^X_{ij} on M so that the bracket identity

$$[X, D_j] = \sum_{i}^{m} C^X_{ij} D_i \tag{1.1}$$

holds. (For the precise definition see for example Section 2 of [2].) Many examples of conformally invariant systems implicitly or explicitly exist in the literature. The Laplacian Δ on $\mathbb{R}^n$ and wave operator $\Box$ on the Minkowski space $\mathbb{R}^{3,1}$ are for instance two outstanding examples for conformally invariant systems consisting of one differential operator. For more examples see for example the introduction of [14].

A project for conformally invariant systems started with the work of Barchini-Kable-Zierau in [1] and [2], and the project was continued in subsequent papers. (For instance [6, 7, 8, 9, 10], [14, 15, 16].) We may want to note that one may find the introduction of [6] helpful to see the recent development of the theory of conformally invariant systems.

The present work is also part of the project. The aim of this paper is to close the cases that were left open in [14] and [16]. To describe our work of this paper more precisely, we now briefly review the works in these papers. Let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Give a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ on $\mathfrak{g}$ so that $\mathfrak{q} = \mathfrak{g}(0) \oplus \bigoplus_{j>0} \mathfrak{g}(j) = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra. Let $Q = N_G(\mathfrak{q}) = LN$. For a real form $\mathfrak{g}_0$ of $\mathfrak{g}$, define G_0 to be an analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Set $Q_0 = N_{G_0}(\mathfrak{q})$. Our manifold is $M = G_0/Q_0$ and we consider a line bundle $\mathcal{L}_s \to G_0/Q_0$ for each $s \in \mathbb{C}$. As the homogeneous space G_0/Q_0 admits

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an open dense submanifold $\bar{N}_0Q_0/Q_0$, we restrict our bundle to this submanifold. By slight abuse of notation we refer to the restricted bundle as $\mathcal{L}_s$. The systems that we shall construct act on smooth sections of the restricted bundle $\mathcal{L}_s \to \bar{N}_0$.

Our systems of operators are constructed from L-irreducible constituents W of $\mathfrak{g}(-r+k)\otimes\mathfrak{g}(r)$ for $1 \leq k \leq 2r$. We call the systems of operators Ω_k *systems.* (We shall describe the construction more precisely in Section 2.2.) There is no reason to expect that Ω_k systems are conformally invariant on $\mathcal{L}_s$ for arbitrary $s \in \mathbb{C}$; the conformal invariance of Ω_k systems depends on the complex parameter s for the line bundle $\mathcal{L}_s$. We then say that an Ω_k system has *special value* s_k if the system is conformally invariant on the line bundle $\mathcal{L}_{s_k}$.

In [14], the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ was taken to be a maximal parabolic subalgebra of quasi-Heisenberg type, and we sought the special values for the Ω_1 system and Ω_2 systems. While the special value s_1 for the Ω_1 system was determined for each parabolic subalgebra $\mathfrak{q}$ as $s_1 = 0$, we left three cases open for Ω_2 systems. To describe the open cases, let us now start explaining briefly the classification for the irreducible constituents W that contribute to Ω_2 systems. In [14], we first observed that if irreducible constituents W contribute to Ω_2 systems then their highest weights are of the form $\mu + \epsilon$, where μ is the highest weight for $\mathfrak{g}(1)$ and ϵ is some weight for $\mathfrak{g}(1)$. We called such irreducible constituents *special* and classified as type 1a, type 1b, type 2, and type 3 with respect to certain technical conditions for the highest weight $\mu + \epsilon$. (We shall give the precise conditions in Definition 2.22.) Table 1 summarizes the types of special constituents. Here, for example, "$B_n(i)$" indicates that the maximal standard parabolic subalgebra of $\mathfrak{g}$ of type B_n, which is determined by the ith simple root α_i. In the table $V(\mu+\epsilon_\gamma)$, $V(\mu+\epsilon_{n\gamma})$, and $V(\mu+\epsilon^{\pm}_{n\gamma})$ denote the special constituents with highest weights $\mu + \epsilon_\gamma$, $\mu + \epsilon_{n\gamma}$, and $\mu + \epsilon^{\pm}_{n\gamma}$, respectively. (We will precisely describe these highest weights in Section 2.4 and Appendix A). As illustrated in Table 1, there are one, two, or three special constituents. A dash in the column for $V(\mu + \epsilon_{n\gamma})$ indicates that there is no special constituent $V(\mu + \epsilon_{n\gamma})$ in the case.

In [14], under the assumption that $\mathfrak{q}$ is not of type $D_n(n-2)$, we found the special values s_2 for Ω_2 systems for the type 1a and type 2 constituents. The technique that we used allowed us to handle each case uniformly. However, since the technique relied on some technical conditions on the highest weights for the special constituents, we could not apply it to the systems coming from type 2 and type 3 constituents.

If $V(\mu+\epsilon) := V(\mu+\epsilon_\gamma)$, $V(\mu+\epsilon_{n\gamma})$, or $V(\mu+\epsilon^{\pm}_{n\gamma})$ then the missing cases may be summarized as follows:

(1) the special constituent $V(\mu+\epsilon)$ is of type 1b,
(2) the special constituent $V(\mu+\epsilon)$ is of type 3, and
(3) the maximal parabolic subalgebra $\mathfrak{q}$ is of type $D_n(n-2)$.

These are the cases boxed in Table 1. Our goal of this paper is to find the special values s_2 of Ω_2 systems in these three cases. To achieve the goal, while we treated each case as uniformly as possible in [14], we in this paper handle these cases separately. For type 1b and type 3 cases, we use an explicit realization of a Lie algebra $\mathfrak{g}$. In this way certain computations can be carried out

TABLE 1. types of special constituents

Parabolic subalgebra	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$B_n(i),\ 3 \leq i \leq n-2$	Type 1a	Type 1a
$B_n(n-1)$	Type 1a	$\boxed{\text{Type 1b}}$
$B_n(n)$	Type 2	–
$C_n(i),\ 2 \leq i \leq n-1$	$\boxed{\text{Type 3}}$	Type 2
$D_n(i),\ 3 \leq i \leq n-3$	Type 1a	Type 1a
$E_6(3)$	Type 1a	Type 1a
$E_6(5)$	Type 1a	Type 1a
$E_7(2)$	Type 1a	–
$E_7(6)$	Type 1a	Type 1a
$E_8(1)$	Type 1a	–
$F_4(4)$	Type 2	–

Parabolic subalgebra	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon^+_{n\gamma})$	$V(\mu+\epsilon^-_{n\gamma})$
$\boxed{D_n(n-2)}$	Type 1a	Type 1a	Type 1a

easily. For the case that $\mathfrak{q}$ is of type $D_n(n-2)$, we first observe that each special constituent is of type 1a. We then apply the technique used in [14].

The special values s_2 for the type 1b, type 3, and type $D_n(n-2)$ cases are $s_2 = 1$ (Theorems 3.11), $s_2 = n-i+1$ (Theorem 4.24), and $s_2 = 1$ (Corollary A.3), respectively. Now, with these results together with ones in [14], if Δ and $\Delta(\mathfrak{g}(1))$ denote a (fixed) root system and the set of roots contributing to $\mathfrak{g}(1)$, respectively, then we obtain the following consequence:

Consequence 1.2. *Let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type. The special value s_2 of the Ω_2 system associated to the special constituent $V(\mu+\epsilon)$ is*

$$s_2 = \begin{cases} \frac{|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|}{2} & \text{if } V(\mu+\epsilon) \text{ is of type 1,} \\ -1 & \text{if } V(\mu+\epsilon) \text{ is of type 2, and} \\ n-i+1 & \text{if } V(\mu+\epsilon) \text{ is of type 3,} \end{cases}$$

where $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|$ is the number of elements of $\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) := \{\alpha \in \Delta(\mathfrak{g}(1)) \mid (\mu+\epsilon)-\alpha \in \Delta\}$.

Here we combine type 1b with type 1a. This is because it turned out that the special value for the type 1b case can be given by the same formula as for the type 1a case. (See Remark 3.12.) If $\Omega_2|_{V(\mu+\epsilon)^*}$ denotes the Ω_2 system coming from the special constituent $V(\mu+\epsilon)$ then Table 2 exhibits the special values for all the Ω_2 systems under consideration. Here λ_i denotes the fundamental weight for the simple root α_i and $\mathcal{L}(s\lambda_i) := \mathcal{L}_s$, the restricted line bundle over $\bar{N}_0$.

Now we turn to [16]. To describe the work of the paper, we first recall that it was shown in [2] that a conformally invariant system yields a homomorphism between appropriate generalized Verma modules, one of which is (in general) non-scalar. A homomorphism between generalized Verma modules is called *standard* if it comes from a homomorphism between corresponding (full) Verma modules, and called *non-standard* otherwise. In [16], we then classified the "standardness" of the homomorphisms φ_{Ω_k} between generalized Verma modules arising from the conformally invariant

Table 2. Line bundles with special values

Parabolic subalgebra	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \leq i \leq n-2$	$\mathcal{L}\big((n-i-\frac{1}{2})\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$B_n(n-1)$	$\mathcal{L}\big(\frac{1}{2}\lambda_{n-1}\big)$	$\mathcal{L}(\lambda_{n-1})$
$B_n(n)$	$\mathcal{L}(-\lambda_n)$	$-$
$C_n(i), 2 \leq i \leq n-1$	$\mathcal{L}\big((n-i+1)\lambda_i\big)$	$\mathcal{L}(-\lambda_i)$
$D_n(i), 3 \leq i \leq n-3$	$\mathcal{L}\big((n-i-1)\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$E_6(3)$	$\mathcal{L}(\lambda_3)$	$\mathcal{L}(2\lambda_3)$
$E_6(5)$	$\mathcal{L}(\lambda_5)$	$\mathcal{L}(2\lambda_5)$
$E_7(2)$	$\mathcal{L}(2\lambda_2)$	$-$
$E_7(6)$	$\mathcal{L}(\lambda_6)$	$\mathcal{L}(3\lambda_6)$
$E_8(1)$	$\mathcal{L}(3\lambda_1)$	$-$
$F_4(4)$	$\mathcal{L}(-\lambda_4)$	$-$

Parabolic subalgebra	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon^+_{n\gamma})^*}$	$\Omega_2\vert_{V(\mu+\epsilon^-_{n\gamma})^*}$
$D_n(n-2)$	$\mathcal{L}(\lambda_{n-2})$	$\mathcal{L}(\lambda_{n-2})$	$\mathcal{L}(\lambda_{n-2})$

Ω_1 and Ω_2 systems constructed in [14]. While the map φ_{Ω_1} was shown to be standard for each parabolic subalgebra $\mathfrak{q}$, as there were several cases in which the special values s_2 for Ω_2 systems were missing, the classification for the map φ_{Ω_2} was not complete. Thus, in this paper, we also determine wether or not the maps φ_{Ω_2} coming from the conformally invariant Ω_2 systems in the missing cases are standard. The classification results are given in Theorem 3.14 (type 1b case), Theorem 4.28 (type 3 case), and Theorem A.6 (type $D_n(n-2)$ case). It turned out that in each case the map φ_{Ω_2} is non-standard. With the results from [16], Table 3 summarizes the classification of the standardness of the maps φ_{Ω_2}.

Table 3. The classification of φ_{Ω_2}

Parabolic subalgebra	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \leq i \leq n-2$	standard	non-standard
$B_n(n-1)$	standard	non-standard
$B_n(n)$	standard	$-$
$C_n(i), 2 \leq i \leq n-1$	non-standard	standard
$D_n(i), 3 \leq i \leq n-3$	non-standard	non-standard
$E_6(3)$	non-standard	non-standard
$E_6(5)$	non-standard	non-standard
$E_7(2)$	non-standard	$-$
$E_7(6)$	non-standard	non-standard
$E_8(1)$	non-standard	$-$
$F_4(4)$	standard	$-$

Parabolic subalgebra	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon^+_{n\gamma})^*}$	$\Omega_2\vert_{V(\mu+\epsilon^-_{n\gamma})^*}$
$D_n(n-2)$	non-standard	non-standard	non-standard

Recall that, for each parabolic subalgebra $\mathfrak{q}$, the special value s_1 for the Ω_1 system is $s_1 = 0$ and that the map φ_{Ω_1} is standard. Now, with the results and the results in Tables 1 and 3, we obtain the following consequence:

Consequence 1.3. *Let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type. The map φ_{Ω_k} for $k = 1, 2$ is non-standard if and only if the special value s_k of the Ω_k system is a positive integer.*

Before closing this introduction we may want to make two remarks on this paper. The first is on the technique to determine the special values of Ω_2 systems. In [14], to determine the special values, we used some reduction techniques. Although the techniques significantly reduced the amount of computations, as several technical formulas on differential operators were used, the computations were still somewhat not straightforward. Now it is known that special values can be obtained by computations in generalized Verma modules. (See for instance [9].) Then, in this paper, by combining the idea used in [14] with that for generalized Verma modules, we are successful to further simplify computations. The technique is given in Proposition 2.23.

The second remark is on the definition of special constituents. As stated in the introduction in [16], there were certain discrepancy on the terminology "special constituents" between [14] and [16]. In [14] (and above of this introduction) we defined the special constituents for Ω_2 systems as ones whose highest weights satisfy certain conditions. On the other hand, in [16], we redefined such constituents for any Ω_k systems as irreducible constituents that contribute to Ω_k systems. The reason why we redefined special constituents as above is that the later definition works not only for Ω_2 systems but also for any Ω_k systems and also that the irreducible constituents with highest weights satisfying the technical conditions are highly expected to contribute to Ω_2 systems. Indeed, in [14], the implication was verified except the three missing cases. In this paper we showed that the implication holds also in the three cases, namely, in Proposition 3.2 (type 1b), Proposition 4.19 (type 3), and Section A.1 (type $D_n(n-2)$ case). Now the two notions of special constituents do agree for Ω_2 systems.

We now outline the rest of this paper. This paper consists of four sections (with this introduction) and two appendices. In Section 2 we review the works of [14] and [16]. In particular we give the precise construction of Ω_k systems. We also review about maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heiseberg type in this section. In Sections 3 and 4, we discuss about the Ω_2 systems arising from the type 1b special constituent and type 3 special constituent, respectively. In these sections we find their special values and determine the standardness of the map φ_{Ω_2}. These are achieved in Theorems 3.10 and 3.14 for type 1b case and Theorems 3.10 and 3.14 for type 3 case, respectively. We handle the case that $\mathfrak{q}$ is of type $D_n(n-2)$ in Appendix A. This is because it turned out that we were able to simply apply the technique from [14] to this case. The special value is given in Corollary A.3 together with Theorem A.2. The standardness of φ_{Ω_2} is determined in Theorem A.6. Finally, in Appendix B, we collect the miscellaneous data that will be helpful for the work of this paper.

2. Preliminaries

The purpose of this section is to summarize the framework established in [14] and [16]. The notation and conventions remain in force in the the rest of this paper.

2.1. A specialization of a vector bundle $\mathcal{V} \to M$.

We start with recalling from [14] the smooth manifold M and the vector bundle $\mathcal{V} \to M$ to work with. This is nothing but the non-compact picture of a degenerate principal series.

Let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Fix a maximal connected solvable subgroup B, and write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ for its Lie algebra with $\mathfrak{h}$ the Cartan subalgebra and $\mathfrak{u}$ the nilpotent subalgebra. Let $\mathfrak{q} \supset \mathfrak{b}$ be a standard parabolic subalgebra of $\mathfrak{g}$. If $Q = N_G(\mathfrak{q})$ then write $Q = LN$ for the Levi decomposition of Q. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ in which the complex parabolic subalgebra $\mathfrak{q}$ has a real form $\mathfrak{q}_0$. Let G_0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Define $Q_0 = N_{G_0}(\mathfrak{q}) \subset Q$, and write $Q_0 = L_0N_0$. We will work with G_0/Q_0 for a class of maximal parabolic subgroup Q_0 whose Lie algebra $\mathfrak{q}_0$ is of quasi-Heisenberg type. (See Section 2.4.)

Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Write Δ^+ for the positive system attached to $\mathfrak{b}$ and denote by Π the set of simple roots. We write $\mathfrak{g}_\alpha$ for the root space for $\alpha \in \Delta$. For each subset $S \subset \Pi$, let $\mathfrak{q}_S$ be the corresponding standard parabolic subalgebra. Write $\mathfrak{q}_S = \mathfrak{l}_S \oplus \mathfrak{n}_S$ with Levi factor $\mathfrak{l}_S = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_S} \mathfrak{g}_\alpha$ and nilpotent radical $\mathfrak{n}_S = \bigoplus_{\alpha\in\Delta^+\backslash\Delta_S} \mathfrak{g}_\alpha$, where $\Delta_S = \{\alpha \in \Delta \mid \alpha \in \operatorname{span}(\Pi\backslash S)\}$. If Q_0 is a maximal parabolic subgroup then there exists a unique simple root $\alpha_{\mathfrak{q}} \in \Pi$ so that $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}}$. Let $\lambda_{\mathfrak{q}}$ be the fundamental weight of $\alpha_{\mathfrak{q}}$. The weight $\lambda_{\mathfrak{q}}$ is orthogonal to any roots α with $\mathfrak{g}_\alpha \subset [\mathfrak{l}, \mathfrak{l}]$. Hence it exponentiates to a character $\chi_{\mathfrak{q}}$ of L. As $\chi_{\mathfrak{q}}$ takes real values on L_0, for $s \in \mathbb{C}$, character $\chi^s = |\chi_{\mathfrak{q}}|^s$ is well-defined on L_0. Let $\mathbb{C}_{\chi^s}$ be the one-dimensional representation of L_0 with character χ^s. The representation χ^s is extended to a representation of Q_0 by making it trivial on N_0. It then deduces a line bundle $\mathcal{L}_s$ on G_0/Q_0 with fiber $\mathbb{C}_{\chi^s}$.

The group G_0 acts on the space

$$C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^s}) = \{F \in C^\infty(G_0, \mathbb{C}_{\chi^s}) \mid F(gq) = \chi^s(q^{-1})F(g) \text{ for all } q \in Q_0 \text{ and } g \in G_0\}$$

by left translation. The action of $\mathfrak{g}_0$ on $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^s})$ arising from this action is given by

$$(Y\bullet F)(g) = \frac{d}{dt}F(\exp(-tY)g)\big|_{t=0} \tag{2.1}$$

for $Y \in \mathfrak{g}_0$, where the dot $\bullet$ denotes the action of the Lie algebra as differential operators. This action is extended $\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Let $\bar{N}_0$ be the unipotent subgroup opposite to N_0. The natural infinitesimal action of $\mathfrak{g}$ on the image of the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^s}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^s})$ induced by (2.1) gives an action of $\mathfrak{g}$ on the whole space $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^s})$. The line bundle $\mathcal{L}_s \to G_0/Q_0$ restricted to $\bar{N}_0$ is the trivial bundle $\bar{N}_0 \times \mathbb{C}_{\chi^s} \to \bar{N}_0$. We shall work with the trivial bundle on $\bar{N}_0$. By slight abuse of notation, we refer to the trivial bundle over $\bar{N}_0$ as $\mathcal{L}_s$.

2.2. The Ω_k systems.

A systematic construction of systems of differential operators for the line bundle $\mathcal{L}_s \to \bar{N}_0$ was established in [14]. In this subsection we summarize the construction of the

systems of differential operators. For a subspace W of $\mathfrak{g}$, we write $\Delta(W) = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset W\}$ and $\Pi(W) = \Delta(W) \cap \Pi$. We keep the notation from the previous subsection, unless otherwise specified.

Let $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ be a $\mathbb{Z}$-grading on $\mathfrak{g}$ with $\mathfrak{g}(1) \neq 0$. Take L to be the analytic subgroup of G with Lie algebra $\mathfrak{g}(0)$. Observe that, as $[\mathfrak{g}(0), \mathfrak{g}(j)] \subset \mathfrak{g}(j)$, each graded subspace $\mathfrak{g}(j)$ is an L-module and so is $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$. Write R for the infinitesimal right translation of $\mathfrak{g}_0$. As usual, we extend it $\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

We build systems of differential operators in three steps as follows:

<u>*Step 1:*</u> First, for $1 \leq k \leq 2r$, consider the L-equivariant polynomial map

$$\begin{aligned} \tau_k : \mathfrak{g}(1) &\to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r) \\ X &\mapsto \big(\mathrm{ad}(X)^k \otimes \mathrm{Id}\big)\omega, \end{aligned}$$

where ω is the element in $\mathfrak{g}(-r) \otimes \mathfrak{g}(r)$ defined by

$$\omega = \sum_{\gamma_j \in \Delta(\mathfrak{g}(r))} X^*_{\gamma_j} \otimes X_{\gamma_j}. \tag{2.2}$$

Here X_{γ_j} are root vectors for γ_j, and $X^*_{\gamma_j}$ are the vectors dual to X_{γ_j} with respect to the Killing form κ, namely, $X^*_{\gamma_j}(X_{\gamma_t}) := \kappa(X^*_{\gamma_j}, X_{\gamma_t}) = \delta_{j,t}$ with $\delta_{i,t}$ the Kronecker delta.

<u>*Step 2:*</u> Next, for an L-irreducible constituent W of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$, consider the associated L-intertwining operator $\tilde{\tau}_k|_{W^*} \in \mathrm{Hom}_L(W^*, \mathcal{P}^k(\mathfrak{g}))$ defined by

$$\tilde{\tau}_k|_{W^*}(Y^*)(X) = Y^*(\tau_k(X)),$$

where W^* is the dual space for W with respect to the Killing form κ. Take an irreducible constituent W of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ so that $\tilde{\tau}_k|_{W^*} \not\equiv 0$.

<u>*Step 3:*</u> Last, to the space W^* dual to the space W taken in Step 2, apply the following algebraic procedure:

$$W^* \stackrel{\tilde{\tau}_k|_{W^*}}{\to} \mathcal{P}^k(\mathfrak{g}(1)) \cong \mathrm{Sym}^k(\mathfrak{g}(-1)) \stackrel{\sigma}{\hookrightarrow} \mathcal{U}(\bar{\mathfrak{n}}) \stackrel{R}{\to} \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}. \tag{2.3}$$

Here, $\sigma : \mathrm{Sym}^k(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ is the symmetrization operator and $\mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ is the space of $\bar{\mathfrak{n}}$-invariant differential operators for $\mathcal{L}_s$.

Let $\Omega_k|_{W^*} : W^* \to \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ be the composition of the linear maps described in (2.3), namely, $\Omega_k|_{W^*} = R \circ \sigma \circ \tilde{\tau}_k|_{W^*}$. For simplicity we write $\Omega_k(Y^*) = \Omega_k|_{W^*}(Y^*)$ for the differential operator arising from $Y^* \in W^*$.

Now, given basis $\{Y_1^*, \ldots, Y_m^*\}$ for W^*, we have a system of differential operators

$$\Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*).$$

We call the system of operators the $\Omega_k|_{W^*}$ **system**. When the irreducible constituent W^* is not important, we simply refer to each $\Omega_k|_{W^*}$ system as an Ω_k system. We may want to note that Ω_k systems are independent of a choice for a basis for W^* up to some natural equivalence. (See Definition 3.5 of [14].)

The Ω_k systems are not conformally invariant for arbitrary complex parameter $s \in \mathbb{C}$ for the line bundle $\mathcal{L}_s$. We then say that an Ω_k system has **special value** s_k if the system is conformally invariant on the line bundle $\mathcal{L}_{s_k}$.

2.3. **The Ω_k systems and generalized Verma modules.** Conformally invariant systems yield non-zero $\mathcal{U}(\mathfrak{g})$-homomorphisms between appropriate generalized Verma modules. Since the theory simplifies computations to find the special values of Ω_k systems, in this subsection, we review how conformally invariant Ω_k systems induce such homomorphisms.

We start with a well-known fact between homogeneous vector bundles and generalized Verma modules. A generalized Verma module $M_{\mathfrak{q}}[E] := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} E$ is a $\mathcal{U}(\mathfrak{g})$-module that is induced from a finite dimensional simple $\mathfrak{q}$-module E. It is well known that if $\mathcal{E} \to G_0/Q_0$ is the homogenous bundle with fiber E then there is a natural pairing between the space $\Gamma(\mathcal{E})$ of smooth sections and generalized Verma module $M_{\mathfrak{q}}[E^*]$, where E^* is the dual space for E. (See for example [4] and [5].) Via this natural pairing, associated to the line bundle $\mathcal{L}_s$ is the generalized Verma module $M_{\mathfrak{q}}[\mathbb{C}_{-s}]$, where $\mathbb{C}_{-s} = \mathbb{C}_{-s\lambda_{\mathfrak{q}}}$ is the $\mathfrak{q}$-module derived from the Q_0-representation $(\chi^{-s}, \mathbb{C})$ with $d\chi = \lambda_{\mathfrak{q}}$.

Now, given irreducible constituent W of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$, we define an L-intertwining operator $\omega_k|_{W^*} : W^* \to \mathcal{U}(\bar{\mathfrak{n}})$ by $\omega_k|_{W^*} = \sigma \circ \tilde{\tau}|_{W^*}$, so that $\Omega_k|_{W^*} = R \circ \omega_k|_{W^*}$. If writing $\omega_k(W^*) = \omega_k|_{W^*}(W^*)$ then we obtain the following diagram:

$$\begin{tikzcd}
 & W^* \arrow[d, "\omega_k|_{W^*}"] & \\
M_{\mathfrak{q}}[\mathbb{C}_{-s}] & \mathcal{U}(\bar{\mathfrak{n}}) \arrow[l, "\cdot\otimes\mathbb{C}_{-s}"'] \arrow[r, "R"] & \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}} \\
\omega_k(W^*) \otimes \mathbb{C}_{-s} & \omega_k(W^*) \arrow[l, mapsto] \arrow[r, squiggly, mapsto] & \{\Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*)\},
\end{tikzcd} \tag{2.4}$$

where $\{Y_1^*, \ldots, Y_m^*\}$ is a given basis for W^*. Here we indicate by the squiggly arrow $\rightsquigarrow$ that the right differentiation R is only applied for the basis elements $\omega_k(Y_1^*), \ldots, \omega_k(Y_m^*)$ in $\omega_k(W^*)$.

It can be seen from (2.4) that constructing the $\Omega_k|_{W^*}$ system is equivalent to making the L-submodule $\omega_k(W^*) \otimes \mathbb{C}_{-s}$. Proposition 2.5 below shows that there is a further relationship.

Proposition 2.5. [2, Theorem 19] *The $\Omega_k|_{W^*}$ system is conformally invariant on the line bundle $\mathcal{L}_{s_k}$ if and only if*

$$\omega_k(W^*) \otimes \mathbb{C}_{-s_k} \subset M_{\mathfrak{q}}[\mathbb{C}_{-s_k}]^{\mathfrak{n}},$$

where $M_{\mathfrak{q}}[\mathbb{C}_{-s}]^{\mathfrak{n}} = \{v \in M_{\mathfrak{q}}[\mathbb{C}_{-s}] \mid X \cdot v = 0 \text{ for all } X \in \mathfrak{n}\}$.

It follows from Proposition 2.5 that if the $\Omega_k|_{W^*}$ system is conformally invariant on $\mathcal{L}_{s_k}$ then the L-submodule $\omega_k(W^*) \otimes \mathbb{C}_{-s_k}$ induces a $\mathcal{U}(\mathfrak{g})$-module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \big(\omega_k(W^*) \otimes \mathbb{C}_{-s_k}\big)$. The following proposition shows that this is indeed a generalized Verma module.

Proposition 2.6. [16, Proposition 3.4 (1)] *If W^* has highest weight ν then $\omega_k(W^*) \otimes \mathbb{C}_{-s}$ is the simple L-submodule of $M_{\mathfrak{q}}[\mathbb{C}_{-s}]$ with highest weight $\nu - s\lambda_{\mathfrak{q}}$.*

Now it follows from Propositions 2.5 and 2.6 that a conformally invariant Ω_k system induces a homomorphism from $M_{\mathfrak{q}}[\omega_k(W^*) \otimes \mathbb{C}_{-s_k}]$ to $M_{\mathfrak{q}}[\mathbb{C}_{-s_k}]$. Indeed, if the $\Omega_k|_{W^*}$ system is conformally invariant for $\mathcal{L}_{s_k}$ then, by Proposition 2.5, there exists inclusion map $\iota \in \mathrm{Hom}_L\big(\omega_k(W^*) \otimes$

$\mathbb{C}_{-s_k}, M_{\mathfrak{q}}[\mathbb{C}_{-s_k}]\big)$. The inclusion map ι then induces a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism

$$\varphi_{\Omega_k} \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}\big(M_{\mathfrak{q}}[\omega_k(W^*)\otimes \mathbb{C}_{-s_k}], M_{\mathfrak{q}}[\mathbb{C}_{-s_k}]\big),$$

that is given by

$$\begin{gathered} M_{\mathfrak{q}}[\omega_k(W^*)\otimes \mathbb{C}_{-s_k}] \stackrel{\varphi_{\Omega_k}}{\to} M_{\mathfrak{q}}[\mathbb{C}_{-s_k}] \\ u\otimes \big(\omega_k(Y^*)\otimes 1\big) \mapsto u\cdot \iota\big(\omega_k(Y^*)\otimes 1\big). \end{gathered} \tag{2.7}$$

Observe that there is a quotient map from a (full) Verma module to a generalized Verma module. A homomorphism between generalized Verma modules is called **standard** if it comes from a homomorphism between the corresponding full Verma modules, and called **non-standard** otherwise ([17]). In the rest of this subsection, to study the standardness of the map φ_{Ω_k} in (2.7), we give a simple criterion to determine whether or not the standard homomorphism $\varphi_{std} : M_{\mathfrak{q}}[\omega_k(W^*)\otimes\mathbb{C}_{-s_k}] \to M_{\mathfrak{q}}[\mathbb{C}_{-s_k}]$ is zero. To do so, it is convenient to parametrize generalized Verma modules by their infinitesimal characters. Therefore we write

$$M_{\mathfrak{q}}[\mathbb{C}_{-s_k\lambda_{\mathfrak{q}}}] = M_{\mathfrak{q}}(-s_k\lambda_{\mathfrak{q}}+\rho), \tag{2.8}$$

where ρ is half the sum of the positive roots. Similarly, if W^* has highest weight ν then, by Proposition 2.6, we write

$$M_{\mathfrak{q}}[\omega_k(W^*)\otimes \mathbb{C}_{-s_k}] = M_{\mathfrak{q}}(\nu - s_k\lambda_{\mathfrak{q}}+\rho). \tag{2.9}$$

Now, if $v := \omega_k(Y^*)\otimes 1$ then (2.7) is expressed by

$$\begin{gathered} M_{\mathfrak{q}}(\nu - s_k\lambda_{\mathfrak{q}}+\rho) \stackrel{\varphi_{\Omega_k}}{\to} M_{\mathfrak{q}}(-s_k\lambda_{\mathfrak{q}}+\rho) \\ u\otimes v \mapsto u\cdot\iota(v). \end{gathered} \tag{2.10}$$

To describe the criterion efficiently we recall a well-known definition for a *link* of two weights. Let $\langle\cdot,\cdot\rangle$ be the inner product on $\mathfrak{h}^*$ induced from the Killing form κ. Write $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$.

Definition 2.11. *(Bernstein-Gelfand-Gelfand) Let $\lambda, \delta \in \mathfrak{h}^*$ and $\beta_1, \ldots, \beta_t \in \Delta^+$. Set $\delta_0 = \delta$ and $\delta_i = s_{\beta_i}\cdots s_{\beta_1}\delta$ for $1 \leq i \leq t$. We say that the sequence $(\beta_1,\ldots,\beta_t)$ **links** δ to λ if*

(1) $\delta_t = \lambda$ and
(2) $\langle \delta_{i-1}, \beta_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq t$.

Let $M(\eta)$ denote the (full) Verma module with highest weight $\eta - \rho$. As usual, if there is a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism from $M(\eta)$ into $M(\zeta)$ then we write $M(\eta) \subset M(\zeta)$. If $\Pi(\mathfrak{l})$ denotes the set of simple roots $\alpha \in \Pi$ so that $\mathfrak{g}_\alpha \subset \mathfrak{l}$ then the criterion is given as follows.

Proposition 2.12. [16, Proposition 4.6] *Let $M_{\mathfrak{q}}(\nu - s_k\lambda_{\mathfrak{q}}+\rho)$ and $M_{\mathfrak{q}}(-s_k\lambda_{\mathfrak{q}}+\rho)$ be the generalized Verma modules in (2.10). Then the standard map from $M_{\mathfrak{q}}(\nu - s_k\lambda_{\mathfrak{q}}+\rho)$ to $M_{\mathfrak{q}}(-s_k\lambda_{\mathfrak{q}}+\rho)$ is zero if and only if there exists $\alpha_\nu \in \Pi(\mathfrak{l})$ so that $-\alpha_\nu - s_k\lambda_{\mathfrak{q}}+\rho$ is linked to $\nu - s_k\lambda_{\mathfrak{q}}+\rho$.*

2.4. **The Ω_2 systems associated to maximal parabolic subalgebras of quasi-Heisenberg type.** In Sections 3 and 4, and Appendix A, we study Ω_2 systems associated to so-called maximal parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ of quasi-Heisenberg type. More precisely we find their special values and determine the standardness of the maps φ_{Ω_2}. Then, in this subsection, we recall from [14] some observation on the Ω_2 systems associated to such maximal parabolic subalgebras.

First, we recall from [14] that a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is called **quasi-Heisenberg type** if its nilpotent radical $\mathfrak{n}$ satisfies the conditions that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$ and $\dim([\mathfrak{n}, \mathfrak{n}]) > 1$. Let $\alpha_\mathfrak{q}$ be a simple root, so that the maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{q}_{\{\alpha_\mathfrak{q}\}} = \mathfrak{l} \oplus \mathfrak{n}$ determined by $\alpha_\mathfrak{q}$ is of quasi-Heisenberg type. Given Dynkin type $\mathcal{T}$ of $\mathfrak{g}$, if we write $\mathcal{T}(i)$ for the Lie algebra together with the choice of maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{q}_{\{\alpha_i\}}$ determined by α_i then the maximal parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ of quasi-Heisenberg type are classified as follows:

$$B_n(i)\ (3 \leq i \leq n), \quad C_n(i)\ (2 \leq i \leq n-1), \quad D_n(i)\ (3 \leq i \leq n-2), \tag{2.13}$$

and

$$E_6(3),\ E_6(5),\ E_7(2),\ E_7(6),\ E_8(1),\ F_4(4). \tag{2.14}$$

Here, the Bourbaki conventions [3] are used for the labels of the simple roots. Note that, in type A_n, any maximal parabolic subalgebra has abelian nilpotent radical, and also that, in type G_2, the two maximal parabolic subalgebras are of either 3-step nilpotent type or Heisenberg type.

We next observe that a maximal parabolic subalgebra $\mathfrak{q}$ of quasi-Heisenberg type induces a 2-grading on $\mathfrak{g}$. As $\mathfrak{q}$ has two-step nilpotent radical, if $\lambda_\mathfrak{q}$ is the fundamental weight for $\alpha_\mathfrak{q}$ then, for all $\beta \in \Delta$, the quotient $2\langle \lambda_\mathfrak{q}, \beta\rangle / ||\alpha_\mathfrak{q}||^2$ takes the values of 0, ± 1, or ± 2. (See for example Section 4.1 of [14].) Therefore, if $H_\mathfrak{q}$ is the element in $\mathfrak{h}$ so that $\beta(H_\mathfrak{q}) = 2\langle \lambda_\mathfrak{q}, \beta\rangle / ||\alpha_\mathfrak{q}||^2$ for all $\beta \in \Delta$, and if $\mathfrak{g}(j)$ is the j-eigenspace of $\mathrm{ad}(H_\mathfrak{q})$ on $\mathfrak{g}$ then the adjoint action of $H_\mathfrak{q}$ induces a 2-grading $\mathfrak{g} = \bigoplus_{j=-2}^{2} \mathfrak{g}(j)$ on $\mathfrak{g}$ with parabolic subalgebra $\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, where $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. The subalgebra $\bar{\mathfrak{n}}$, the nilpotent radical opposite to $\mathfrak{n}$, is given by $\bar{\mathfrak{n}} = \mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$. Here we have $\mathfrak{g}(2) = \mathfrak{z}(\mathfrak{n})$ and $\mathfrak{g}(-2) = \mathfrak{z}(\bar{\mathfrak{n}})$, where $\mathfrak{z}(\mathfrak{n})$ (resp. $\mathfrak{z}(\bar{\mathfrak{n}})$) is the center of $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$). Thus we denote the 2-grading on $\mathfrak{g}$ by

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus \mathfrak{g}(-1) \oplus \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n}) \tag{2.15}$$

with parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n}).$$

Now, for $1 \leq k \leq 4$, the maps τ_k associated to the grading (2.15) are given by

$$\begin{aligned} \tau_k : \mathfrak{g}(1) &\to \mathfrak{g}(-2+k) \otimes \mathfrak{z}(\mathfrak{n}) \\ X &\mapsto \frac{1}{k!}\big(\mathrm{ad}(X)^k \otimes \mathrm{Id}\big)\omega \end{aligned} \tag{2.16}$$

with

$$\omega = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} X^*_{\gamma_j} \otimes X_{\gamma_j}. \tag{2.17}$$

In particular when $k = 2$, we have

$$\begin{aligned} \tau_2 : \mathfrak{g}(1) &\to \mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) \\ X &\mapsto \frac{1}{2}\big(\mathrm{ad}(X)^2 \otimes \mathrm{Id}\big)\omega. \end{aligned} \tag{2.18}$$

The Ω_2 systems, the systems of differential operators that we study, are constructed from the τ_2 map.

By construction the Ω_2 systems arise from irreducible constituents W of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ so that $\tilde{\tau}_2|_{W^*}$ is not identically zero. (See the procedures described in Subsection 2.2.) Since such irreducible constituents play a role to determine the special value of the Ω_2 systems, for the remainder of this section, we recall from [14] the observation on $\mathfrak{l}$ and irreducible constituents W^* with $\tilde{\tau}_2|_{W^*} \not\equiv 0$.

We start with the structure of $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$, where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. First, observe that $\mathfrak{z}(\mathfrak{l})$ is one-dimensional. Indeed, we have $\mathfrak{z}(\mathfrak{l}) = \bigcap_{\alpha \in \Pi(\mathfrak{l})} \ker(\alpha)$ with $\Pi(\mathfrak{l}) = \Pi \backslash \{\alpha_{\mathfrak{q}}\}$. As $\mathfrak{l} = \mathfrak{g}(0)$, $\alpha(H_{\mathfrak{q}}) = 0$ for all $\alpha \in \Delta(\mathfrak{l})$. Thus $H_{\mathfrak{q}}$ is an element of $\mathfrak{z}(\mathfrak{l})$, and so we have $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}H_{\mathfrak{q}}$.

To observe the semisimple part $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$, let γ be the highest root for $\mathfrak{g}$. If $\mathfrak{g}$ is not of type A_n then there is exactly one simple root that is not orthogonal to γ. If α_γ is the unique simple root then $\mathfrak{q}' = \mathfrak{q}_{\{\alpha_\gamma\}}$ is the parabolic subalgebra of Heisenberg type, that is, its nilpotent radical $\mathfrak{n}'$ satisfies $\dim([\mathfrak{n}', \mathfrak{n}']) = 1$. Hence, if $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}}$ is a parabolic subalgebra of quasi-Heisenberg type then α_γ is in $\Pi(\mathfrak{l}) = \Pi\backslash\{\alpha_{\mathfrak{q}}\}$. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ is either simple or the direct sum of two or three simple ideals with only one simple ideal containing the root space $\mathfrak{g}_{\alpha_\gamma}$ for α_γ. We denote by $\mathfrak{l}_\gamma$ the unique simple ideal containing $\mathfrak{g}_{\alpha_\gamma}$. Similarly, when $[\mathfrak{l}, \mathfrak{l}]$ consists of two (resp. three) simple ideals, we denote the other simple ideas(s) by $\mathfrak{l}_{n\gamma}$ (resp. $\mathfrak{l}^+_{n\gamma}$ and $\mathfrak{l}^-_{n\gamma}$). The three simple factors occur only when $\mathfrak{q}$ is of type $D_n(n-2)$. So, when $\mathfrak{q}$ is not type $D_n(n-2)$, the Levi subalgebra $\mathfrak{l}$ may decompose into

$$\mathfrak{l} = \mathbb{C}H_{\mathfrak{q}} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}_{n\gamma}. \tag{2.19}$$

Similarly, when $\mathfrak{q}$ is of type $D_n(n-2)$, one may write

$$\mathfrak{l} = \mathbb{C}H_{\mathfrak{q}} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}^+_{n\gamma} \oplus \mathfrak{l}^-_{n\gamma}. \tag{2.20}$$

Note that when $[\mathfrak{l}, \mathfrak{l}]$ is a simple ideal, we have $\mathfrak{l}_{n\gamma} = \{0\}$ ($\mathfrak{l}^{\pm}_{n\gamma} = \{0\}$). It follows from the decompositions (2.19) and (2.20) that the tensor product $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ may be written as

$$\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) = \begin{cases} \big(\mathbb{C}H_{\mathfrak{q}} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big) & \text{if } \mathfrak{q} \text{ is not of type } D_n(n-2) \\ \big(\mathbb{C}H_{\mathfrak{q}} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}^+_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}^-_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big) & \text{if } \mathfrak{q} \text{ is of type } D_n(n-2). \end{cases}$$

To build an Ω_2 system, it is necessary to choose an irreducible constituent W in $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ so that the L-intertwining map

$$\tilde{\tau}_2|_{W^*} \in \mathrm{Hom}_L(W^*, \mathcal{P}^2(\mathfrak{g}(1)))$$

is not identically zero. Now we give a necessary condition for irreducible constituents W so that $\tilde{\tau}_2|_{W^*} \not\equiv 0$. To do so, for $\nu \in \mathfrak{h}^*$ with $\langle \nu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\Pi(\mathfrak{l})$, we denote by $V(\nu)$ the simple $\mathfrak{l}$-module with highest weight $\nu|_{\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]}$.

Suppose that $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ has irreducible constituent $V(\nu)$. If the linear map $\tilde{\tau}_2|_{V(\nu)^*} : V(\nu)^* \to \mathcal{P}^2(\mathfrak{g}(1))$ is not identically zero then, via the isomorphism $\mathcal{P}^2(\mathfrak{g}(1)) \cong \mathrm{Sym}^2(\mathfrak{g}(1))^*$, $V(\nu)$ should be an irreducible constituent of $\mathrm{Sym}^2(\mathfrak{g}(1)) \subset \mathfrak{g}(1) \otimes \mathfrak{g}(1)$. In particular if μ is the highest weight of $\mathfrak{g}(1)$ then ν is of the form $\nu = \mu + \epsilon$ for some $\epsilon \in \Delta(\mathfrak{g}(1))$. It was shown in Lemma 4.14 of [14] that the highest root γ is of this form. However, it follows from Proposition 6.5 of [14] that $V(\gamma)$ does not occur in $\mathrm{Sym}^2(\mathfrak{g}(1))$. Based on this observation we give the following definition.

Definition 2.21. [14, Definition 6.7] *An irreducible constituent $V(\nu)$ of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ is called* ***special***[1] *if $V(\nu)$ satisfies the following two conditions:*

(C1) $\nu = \mu + \epsilon$ *for some* $\epsilon \in \Delta(\mathfrak{g}(1))$.
(C2) $\nu \neq \gamma$.

It is shown in Section 6 of [14] that, for $\mathfrak{q}$ not of type $D_n(n-2)$, there are exactly one or two special constituents of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$; one is an irreducible constituent of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and the other is equal to $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. We denote by $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ the special constituents so that $V(\mu+\epsilon_\gamma) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and $V(\mu+\epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. It will be shown in Appendix A that if $\mathfrak{q}$ is of type $D_n(n-2)$ then there are three special constituents, namely, $V(\mu+\epsilon_\gamma) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and $V(\mu+\epsilon^{\pm}_{n\gamma}) = \mathfrak{l}^{\pm}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$.

To compute the special values of Ω_2 systems efficiently, the special constituents $V(\mu+\epsilon)$ are classified as type 1a, type 1b, type 2, or type 3 as follows:

Definition 2.22. [14, Definition 6.20] *Let μ be the highest weight for $\mathfrak{g}(1)$. We say that a special constituent $V(\mu+\epsilon)$ is of*

1. ***type 1a*** *if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and both μ and ϵ are long roots,*
2. ***type 1b*** *if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and either μ or ϵ is a short root,*
3. ***type 2*** *if $\mu+\epsilon = 2\mu$ is not a root, or*
4. ***type 3*** *if $\mu+\epsilon$ is a root.*

Table 1 in the introduction shows the types of special constituents. A dash in the column for $V(\mu+\epsilon_{n\gamma})$ indicates that $\mathfrak{l}_{n\gamma} = \{0\}$ for the case. (So there is no special constituent $V(\mu+\epsilon_{n\gamma})$.)

In Sections 3 and 4, we find the special values of the Ω_2 systems coming from special constituents of type 1b and type 3, respectively. The following proposition will play a key role to determine the special values.

Proposition 2.23. *Let $V(\mu+\epsilon)^*$ be the dual module of an irreducible constituent $V(\mu+\epsilon)$ of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ so that the operator $\Omega_2|_{V(\mu+\epsilon)^*} : V(\mu+\epsilon)^* \to \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ is non-zero. If X_h is a highest weight vector for $\mathfrak{g}(1)$ and if Y_l^* is a lowest weight vector for $V(\mu+\epsilon)^*$ then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if in $M_{\mathfrak{q}}[\mathbb{C}_{-s}]$*

$$X_\mu \cdot \big(\omega_2(Y_l^*) \otimes 1_{-s}\big) = 0. \tag{2.24}$$

[1]There is a certain discrepancy on the terminology "special constituents". See the comments in the introduction on this matter.

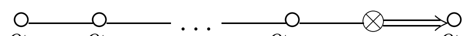
Proof. Observe that, by Proposition 2.5, the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant if and only if, for all $X \in \mathfrak{n}$ and $Y^* \in V(\mu+\epsilon)^*$,

$$X \cdot \big(\omega_2(Y^*) \otimes 1_{-s}\big) = X\omega_2(Y^*) \otimes 1_{-s} = 0. \tag{2.25}$$

Therefore, to prove this proposition, it suffices to show that (2.24) implies (2.25). Since the arguments are similar to ones for Proposition 7.13 with Lemma 3.9 and Lemma 3.12 of [14], we omit the proof. $\square$

3. Type 1b special constituent

In this section we study the Ω_2 system associated to the type 1b special constituent. It follows from Table 1 in the introduction that the type 1b special constituent occurs only when the parabolic subalgebra $\mathfrak{q}$ is of type $B_n(n-1)$. The simple root $\alpha_\mathfrak{q}$ that determines the parabolic subalgebra $\mathfrak{q}$ is then $\alpha_\mathfrak{q} = \alpha_{n-1}$. We write $\lambda_\mathfrak{q} = \lambda_{n-1}$ for the fundamental weight $\lambda_\mathfrak{q}$ for $\alpha_\mathfrak{q}$. The deleted Dynkin diagram for $\mathfrak{q}$ is



with connected subgraphs

(For the definition of deleted Dynkin diagrams see for instance Section 4.1 of [14].) Since α_2 is the unique simple root that is not orthogonal to the highest root γ, it follows from the subgraphs that $\mathfrak{l}_\gamma \cong \mathfrak{sl}(n-1,\mathbb{C})$ and $\mathfrak{l}_{n\gamma} \cong \mathfrak{sl}(2,\mathbb{C})$. Recall from Table 1 that the type 1b special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ is the irreducible constituent $V(\mu+\epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$.

3.1. The $\tilde{\tau}_2|_{V(\mu+\epsilon_{n\gamma})^*}$ map.

We start with observing the L-intertwining operator $\tilde{\tau}_2|_{V(\mu+\epsilon_{n\gamma})^*}$: $V(\mu+\epsilon_{n\gamma})^* \to \mathcal{P}^2(\mathfrak{g}(1))$. To do so we first fix convenient root vectors for $\mathfrak{g}$ so that certain computations will be carried out easily. Observe that, as $\mathfrak{q}$ is of type $B_n(n-1)$, the Lie algebra $\mathfrak{g}$ under consideration is $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$. We take $\mathfrak{h}$ to be the set of block diagonal matrices

$$H(h_1,\ldots,h_n) = \mathrm{diag}\left(\begin{pmatrix} 0 & \mathbf{i}h_1 \\ -\mathbf{i}h_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{i}h_2 \\ -\mathbf{i}h_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \mathbf{i}h_n \\ -\mathbf{i}h_n & 0 \end{pmatrix}, 0 \right)$$

with $h_j \in \mathbb{C}$ and $\mathbf{i} = \sqrt{-1}$. The positive roots are $\Delta^+ = \{\varepsilon_j \pm \varepsilon_k \mid 1 \le j < k \le n\} \cup \{\varepsilon_j \mid 1 \le j \le n\}$ with $\varepsilon_j(H(h_1,\ldots,h_n)) = h_j$. We take the root vectors X_α as follows:

(1) $\alpha = \pm(\varepsilon_j \pm \varepsilon_k)$ $(j<k)$:

$$X_\alpha = \begin{array}{cc} \begin{array}{cc} j & k \end{array} & \\ \begin{pmatrix} 0 & E_\alpha \\ -E_\alpha^t & 0 \end{pmatrix} & \begin{array}{c} j \\ k \end{array} \end{array}$$

TABLE 4. The values of $N_{\alpha,\beta}$ for roots α and β for $\mathfrak{so}(2n+1,\mathbb{C})$ with indices $i<j<k$ when $\alpha+\beta$ is a positive root

Formula	α	β	$N_{\alpha,\beta}$
(1)	$\varepsilon_i+\varepsilon_k$	$\varepsilon_j-\varepsilon_k$	-1
(2)	$\varepsilon_i-\varepsilon_k$	$\varepsilon_j+\varepsilon_k$	-1
(3)	$\varepsilon_i+\varepsilon_k$	$-\varepsilon_j-\varepsilon_k$	$+1$
(4)	$\varepsilon_i-\varepsilon_k$	$-\varepsilon_j+\varepsilon_k$	$+1$
(5)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_j+\varepsilon_k$	-1
(6)	$\varepsilon_i-\varepsilon_j$	$\varepsilon_j+\varepsilon_k$	$+1$
(7)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_j-\varepsilon_k$	-1
(8)	$\varepsilon_i-\varepsilon_j$	$\varepsilon_j-\varepsilon_k$	$+1$
(9)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_i+\varepsilon_k$	$+1$
(10)	$-\varepsilon_i+\varepsilon_j$	$\varepsilon_i+\varepsilon_k$	$+1$
(11)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_i-\varepsilon_k$	$+1$
(12)	$-\varepsilon_i+\varepsilon_j$	$\varepsilon_i-\varepsilon_k$	$+1$
(13)	ε_j	$-\varepsilon_j+\varepsilon_k$	-1
(14)	$-\varepsilon_j$	$\varepsilon_j+\varepsilon_k$	-1
(15)	ε_k	$\varepsilon_j-\varepsilon_k$	-1
(16)	$-\varepsilon_k$	$\varepsilon_j+\varepsilon_k$	$+1$
(17)	ε_j	ε_k	-1
(18)	ε_j	$-\varepsilon_k$	$+1$

with

$$E_{\varepsilon_j-\varepsilon_k}=\frac{1}{2}\begin{pmatrix}1 & \mathbf{i}\\ -\mathbf{i} & 1\end{pmatrix},\quad E_{\varepsilon_j+\varepsilon_k}=\frac{1}{2}\begin{pmatrix}1 & -\mathbf{i}\\ -\mathbf{i} & -1\end{pmatrix}$$
$$E_{-\varepsilon_j+\varepsilon_k}=\frac{1}{2}\begin{pmatrix}-1 & \mathbf{i}\\ -\mathbf{i} & -1\end{pmatrix},\quad E_{-\varepsilon_j-\varepsilon_k}=\frac{1}{2}\begin{pmatrix}-1 & -\mathbf{i}\\ -\mathbf{i} & 1\end{pmatrix},$$

where X_α denotes the matrix whose entries are all zero except the jth and kth pairs of indices.

(2) $\alpha=\pm\varepsilon_j$:

$$X_\alpha=\begin{array}{cc} \begin{array}{cc} j & 2n+1 \end{array} & \\ \begin{pmatrix} 0 & v_\alpha \\ -v_\alpha^t & 0 \end{pmatrix} & \begin{array}{c} j \\ 2n+1 \end{array} \end{array}$$

with

$$v_{\varepsilon_j}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\mathbf{i}\end{pmatrix},\quad v_{-\varepsilon_j}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ -\mathbf{i}\end{pmatrix},$$

where X_α denotes the matrix defined similarly to the previous case.

For $\alpha,\beta\in\Delta$ with $\alpha+\beta\neq 0$, let $N_{\alpha,\beta}$ denote the constant so that $[X_\alpha,X_\beta]=N_{\alpha,\beta}X_{\alpha+\beta}$. Table 4 summarizes the values of $N_{\alpha,\beta}$ for $\alpha+\beta$ a positive root. The constant $N_{\alpha,\beta}$ satisfies the property that $N_{-\alpha,-\beta}=-N_{\alpha,\beta}$. (See for instance Theorem 6.6 in [12].) Thus the values of $N_{\alpha,\beta}$ for $\alpha+\beta$ a negative root can also be obtained from Table 4. Here, we would like to acknowledge that for the cases for α, β long roots (Formulas (1)-(12)), we simply adapt Knapp's beautiful multiplication table in Section 10 of [13].

For $\alpha \in \Delta^+$ we set $H_\alpha = [X_\alpha, X_{-\alpha}]$; namely, we have

$$H_{\varepsilon_j \pm \varepsilon_k} = \mathrm{diag}\Big(0, \ldots, 0, \overset{j}{\begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}}, 0, \ldots, 0, \pm \overset{k}{\begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}}, 0, \ldots, 0\Big)$$

and

$$H_{\varepsilon_j} = \mathrm{diag}\Big(0, \ldots, 0, \overset{j}{\begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}}, 0, \ldots, 0, \Big).$$

Now observe if $T(X,Y) := \mathrm{Tr}(XY)$ then $T(X_\alpha, X_{-\alpha}) = 2$ for all $\alpha \in \Delta^+$. As the restriction $T(\cdot,\cdot)|_{\mathfrak{h}_0 \times \mathfrak{h}_0}$ to a real form $\mathfrak{h}_0$ of $\mathfrak{h}$ is an inner product on $\mathfrak{h}_0$, the trace form $T(\cdot,\cdot)$ is a positive constant multiple of the Killing form $\kappa(\cdot,\cdot)$. If b_o is the non-zero constant so that $\kappa(X,Y) = b_o T(X,Y)$ then $\kappa(X_\alpha, X_{-\alpha}) = 2b_o$ for all $\alpha \in \Delta^+$. Thus the dual vectors X_α^* for X_α with respect to the Killing form are $X_\alpha^* = (1/(2b_o))X_{-\alpha}$.

Now the element ω in (2.17) is given by

$$\omega = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{\gamma_j}^* \otimes X_{\gamma_j} = \frac{1}{2b_o} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{-\gamma_j} \otimes X_{\gamma_j}.$$

Thus the map $\tau_2(X)$ in (2.18) may be expressed as

$$\tau_2(X) = \frac{1}{2}\big(\mathrm{ad}(X)^2 \otimes \mathrm{Id}\big)\omega = \frac{1}{4b_o} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{ad}(X)^2 X_{-\gamma_j} \otimes X_{\gamma_j}. \tag{3.1}$$

Proposition 7.3 of [14] showed that when $\mathfrak{q}$ is of type $B_n(n-1)$, the τ_2 map is not identically zero.

To construct differential operators from $V(\mu + \epsilon_{n\gamma})^*$, it is necessary to show that the linear map $\tilde{\tau}_2|_{V(\mu+\epsilon_{n\gamma})^*} : V(\mu + \epsilon_{n\gamma})^* \to \mathcal{P}^2(\mathfrak{g}(1))$ is not identically zero. (See *Step 2* and *Step 3* in Section 2.2.) We shall prove this by showing that $\tilde{\tau}_2(Y^*)(X)$ is a non-zero polynomial on $\mathfrak{g}(1)$ for some Y^* in $V(\mu + \epsilon_{n\gamma})^*$. To this end observe that, as discussed in Section 2.4, we have $V(\mu + \epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. Therefore $V(\mu + \epsilon_{n\gamma})^* = \mathfrak{l}_{n\gamma}^* \otimes \mathfrak{z}(\mathfrak{n})^* = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\bar{\mathfrak{n}})$. Since $\gamma \in \Delta(\mathfrak{z}(\mathfrak{n}))$ and $\mathfrak{l}_{n\gamma} = \mathrm{span}_{\mathbb{C}}\{X_{\alpha_n}, H_{\alpha_n}, X_{-\alpha_n}\}$, we have $X_{\alpha_n}^* \in \mathfrak{l}_{n\gamma}$ and $X_\gamma^* \in \mathfrak{z}(\bar{\mathfrak{n}})$; therefore $X_{\alpha_n}^* \otimes X_\gamma^* \in \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\bar{\mathfrak{n}}) = V(\mu + \epsilon_{n\gamma})^*$.

Proposition 3.2. *If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is the parabolic subalgebra of type $B_n(n-1)$ then the L-intertwining operator $\tilde{\tau}_2|_{V(\mu+\epsilon_{n\gamma})^*}$ is not identically zero.*

Proof. We show that $\tilde{\tau}_2(X_{\alpha_n}^* \otimes X_\gamma^*)(X)$ is a non-zero polynomial on $\mathfrak{g}(1)$. As X_α^* denotes the dual vector for X_α with respect to the Killing form κ, by (3.1), the operator $\tilde{\tau}_2(X_{\alpha_n}^* \otimes X_\gamma^*)(X) =$

$(X^*_{\alpha_n} \otimes X^*_\gamma)(\tau_2(X))$ is given by

$$\begin{aligned}
\tilde{\tau}_2(X^*_{\alpha_n} \otimes X^*_\gamma)(X) &= (X^*_{\alpha_n} \otimes X^*_\gamma)(\tau_2(X)) \\
&= \frac{1}{4b_o} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \kappa(X^*_{\alpha_n}, \mathrm{ad}(X)^2 X_{-\gamma_j})\kappa(X^*_\gamma, X_{\gamma_j}) \\
&= \frac{1}{4b_o}\kappa(X^*_{\alpha_n}, \mathrm{ad}(X)^2 X_{-\gamma}) \\
&= \frac{1}{8b_o^2}\kappa(X_{-\alpha_n}, \mathrm{ad}(X)^2 X_{-\gamma}). \qquad (3.3)
\end{aligned}$$

Write $X = \sum_{\alpha \in \Delta(\mathfrak{g}(1))} \eta_\alpha X_\alpha$, where $\eta_\alpha \in \mathfrak{n}^*$ is the coordinate dual to X_α with respect to the Killing form. If $W \subset \mathfrak{g}$ is an $\mathrm{ad}(\mathfrak{h})$-invariant subspace then, for any weight $\nu \in \mathfrak{h}^*$, we write

$$\Delta_\nu(W) := \{\alpha \in \Delta(W) \mid \nu - \alpha \in \Delta\}. \qquad (3.4)$$

Then,

$$\begin{aligned}
(3.3) &= \frac{1}{8b_o^2}\kappa(X_{-\alpha_n}, \mathrm{ad}(X)^2 X_{-\gamma}) \\
&= \frac{1}{8b_o^2} \sum_{\beta,\delta \in \Delta(\mathfrak{g}(1))} \eta_\beta \eta_\delta \kappa(X_{-\alpha_n}, [X_\delta, [X_\beta, X_{-\gamma}]]) \\
&= \frac{1}{8b_o^2} \sum_{\alpha,\beta \in \Delta(\mathfrak{g}(1))} \eta_\beta \eta_\delta \kappa([X_{-\alpha_n}, X_\delta], [X_\beta, X_{-\gamma}]]) \\
&= \frac{1}{8b_o^2} \sum_{\substack{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \\ \delta \in \Delta_{\alpha_n}(\mathfrak{g}(1))}} \eta_\beta \eta_\delta N_{\beta,-\gamma} N_{-\alpha_n,\delta} \kappa(X_{\delta-\alpha_n}, X_{\beta-\gamma}). \qquad (3.5)
\end{aligned}$$

One may observe that $\kappa(X_{\delta-\alpha_n}, X_{\beta-\gamma}) \neq 0$ if and only if $\delta = \alpha_n + \gamma - \beta \in \Delta(\mathfrak{g}(1))$. If we write

$$\theta(\beta) = \alpha_n + \gamma - \beta \qquad (3.6)$$

then

$$\begin{aligned}
(3.5) &= \frac{1}{8b_o^2} \sum_{\substack{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \\ \delta \in \Delta_{\alpha_n}(\mathfrak{g}(1))}} \eta_\beta \eta_\delta N_{\beta,-\gamma} N_{-\alpha_n,\delta} \kappa(X_{\delta-\alpha_n}, X_{\beta-\gamma}) \\
&= \frac{1}{8b_o^2} \sum_{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \cap \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} \eta_\beta \eta_{\theta(\beta)} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X_{\theta(\beta)-\alpha_n}, X_{\beta-\gamma}) \\
&= \frac{1}{8b_o^2} \sum_{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \cap \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} \eta_\beta \eta_{\theta(\beta)} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X_{\gamma-\beta}, X_{\beta-\gamma}) \\
&= \frac{1}{4b_o} \sum_{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \cap \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} \eta_\beta \eta_{\theta(\beta)} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X^*_{\beta-\gamma}, X_{\beta-\gamma}) \\
&= \frac{1}{4b_o} \sum_{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \cap \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} \eta_\beta \eta_{\theta(\beta)} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \\
&= \frac{1}{4b_o} \sum_{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \cap \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X, X^*_\beta) \kappa(X, X^*_{\theta(\beta)}). \qquad (3.7)
\end{aligned}$$

As $\alpha_n = \varepsilon_n$ and $\gamma = \varepsilon_1 + \varepsilon_2$, by inspection, we have

$$\Delta_\gamma(\mathfrak{g}(1)) = \{\varepsilon_1 \pm \varepsilon_n, \varepsilon_2 \pm \varepsilon_n\} \cup \{\varepsilon_1, \varepsilon_2\} \quad \text{and} \quad \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1)) = \{\varepsilon_1 + \varepsilon_n, \varepsilon_2 + \varepsilon_n\} \cup \{\varepsilon_1, \varepsilon_2\}.$$

In particular, $\Delta_{\alpha_n+\gamma}(\mathfrak{g}(1)) \subset \Delta_\gamma(\mathfrak{g}(1))$. Therefore,

$$\begin{aligned}
(3.7) &= \frac{1}{4b_o} \sum_{\beta \in \Delta_\gamma(\mathfrak{g}(1)) \cap \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X, X^*_\beta)\kappa(X, X^*_{\theta(\beta)}) \\
&= \frac{1}{4b_o} \sum_{\beta \in \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X, X^*_\beta)\kappa(X, X^*_{\theta(\beta)}).
\end{aligned}$$

Therefore we obtain

$$\tilde{\tau}_2(X^*_{\alpha_n} \otimes X^*_\gamma)(X) = \frac{1}{4b_o} \sum_{\beta \in \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X, X^*_\beta)\kappa(X, X^*_{\theta(\beta)}).$$

Since $\Delta_{\alpha_n+\gamma}(\mathfrak{g}(1)) = \{\varepsilon_1 + \varepsilon_n, \varepsilon_2 + \varepsilon_n\} \cup \{\varepsilon_1, \varepsilon_2\}$, this reads

$$\begin{aligned}
&\tilde{\tau}_2(X^*_{\alpha_n} \otimes X^*_\gamma)(X) \\
&= \frac{1}{4b_o} \sum_{\beta \in \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1))} N_{\beta,-\gamma} N_{-\alpha_n,\theta(\beta)} \kappa(X, X^*_\beta)\kappa(X, X^*_{\theta(\beta)}) \qquad (3.8) \\
&= \frac{1}{4b_o}\big(N_{\varepsilon_1+\varepsilon_n,-(\varepsilon_1+\varepsilon_2)} N_{-\varepsilon_n,\varepsilon_2} \kappa(X, X^*_{\varepsilon_1+\varepsilon_n})\kappa(X, X^*_{\varepsilon_2}) + N_{\varepsilon_2,-(\varepsilon_1+\varepsilon_2)} N_{-\varepsilon_n,\varepsilon_1+\varepsilon_n} \kappa(X, X^*_{\varepsilon_2})\kappa(X, X^*_{\varepsilon_1+\varepsilon_n}) \\
&\quad + N_{\varepsilon_2+\varepsilon_n,-(\varepsilon_1+\varepsilon_2)} N_{-\varepsilon_n,\varepsilon_1} \kappa(X, X^*_{\varepsilon_2+\varepsilon_n})\kappa(X, X^*_{\varepsilon_1}) + N_{\varepsilon_1,-(\varepsilon_1+\varepsilon_2)} N_{-\varepsilon_n,\varepsilon_2+\varepsilon_n} \kappa(X, X^*_{\varepsilon_1})\kappa(X, X^*_{\varepsilon_2+\varepsilon_n})\big) \\
&= \frac{1}{4b_o}\big((1)(-1)\kappa(X, X^*_{\varepsilon_1+\varepsilon_n})\kappa(X, X^*_{\varepsilon_2}) + (-1)(1)\kappa(X, X^*_{\varepsilon_2})\kappa(X, X^*_{\varepsilon_1+\varepsilon_n}) \\
&\quad + (-1)(-1)\kappa(X, X^*_{\varepsilon_2+\varepsilon_n})\kappa(X, X^*_{\varepsilon_1})(1)(1)\kappa(X, X^*_{\varepsilon_1})\kappa(X, X^*_{\varepsilon_2+\varepsilon_n})\big) \\
&= \frac{1}{2b_o}\big(\kappa(X, X^*_{\varepsilon_2+\varepsilon_n})\kappa(X, X^*_{\varepsilon_1}) - \kappa(X, X^*_{\varepsilon_1+\varepsilon_n})\kappa(X, X^*_{\varepsilon_2})\big). \qquad (3.9)
\end{aligned}$$

Therefore $\tilde{\tau}_2(X^*_{\alpha_n} \otimes X^*_\gamma)(X)$ is a non-zero polynomial on $\mathfrak{g}(1)$. □

3.2. **The special value of the** $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ **system.** Now we find the special value of the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system. To do so we use Proposition 2.23. To this end recall from Section 2.3 the linear map $\omega_2|_{V(\mu+\epsilon_{n\gamma})^*} : V(\mu+\epsilon_{n\gamma})^* \to \mathcal{U}(\bar{\mathfrak{n}})$ defined by $\omega_2|_{V(\mu+\epsilon_{n\gamma})^*} = \sigma \circ \tilde{\tau}_2|_{V(\mu+\epsilon_{n\gamma})^*}$, where $\sigma : \mathrm{Sym}^2(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ the symmetrization operator. (Here we identify $\mathcal{P}^2(\mathfrak{g}(1)) \cong \mathrm{Sym}^2(\mathfrak{g}(-1))$.) If

$$Y^*_l := 8b_o^3(X^*_{\alpha_n} \otimes X^*_\gamma)$$

then, by (3.8), $\omega_2(Y^*_l) := \omega_2|_{V(\mu+\epsilon_{n\gamma})^*}(Y^*_l)$ is given by

$$\omega_2(Y^*_l) = 4b_o^2\big(\sigma(X^*_{\varepsilon_2+\varepsilon_n} X^*_{\varepsilon_1}) - \sigma(X^*_{\varepsilon_1+\varepsilon_n} X^*_{\varepsilon_2})\big).$$

As the dual vector X^*_α for X_α with respect to the Killing form is $X^*_\alpha = (1/2b_o)X_{-\alpha}$, this amounts to

$$\omega_2(Y^*_l) = \sigma(X_{-(\varepsilon_2+\varepsilon_n)} X_{-\varepsilon_1}) - \sigma(X_{-(\varepsilon_1+\varepsilon_n)} X_{-\varepsilon_2}).$$

Moreover, since $\varepsilon_1+\varepsilon_2+\varepsilon_n \notin \Delta$, the symmetrization is unnecessary. Therefore we obtain

$$\omega_2(Y_l^*) = X_{-(\varepsilon_2+\varepsilon_n)}X_{-\varepsilon_1} - X_{-(\varepsilon_1+\varepsilon_n)}X_{-\varepsilon_2}. \tag{3.10}$$

Now we are going to determine the special value of the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system.

Theorem 3.11. *Let $\mathfrak{q}$ be the maximal parabolic subalgebra of type $B_n(n-1)$. The $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if $s=1$.*

Proof. Observe that, as $X_{\alpha_n}\otimes X_\gamma$ is a highest weight vector for $\mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n}) = V(\mu+\epsilon_{n\gamma})$, $X_{\alpha_n}^*\otimes X_\gamma^*$ is a lowest weight vector for $V(\mu+\epsilon_{n\gamma})^*$; consequently, Y_l^* is a lowest weight vector for $V(\mu+\epsilon_{n\gamma})^*$. Therefore, by Proposition 2.23, to find the special value for the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system, it suffices to determine $s\in\mathbb{C}$ so that $X_\mu\cdot(\omega_2(Y_l^*)\otimes 1_{-s}) = 0$ for μ the highest weight for $\mathfrak{g}(1)$. By inspection, we have $\mu = \varepsilon_1+\varepsilon_n$. (See Appendix B.) Thus we compute $X_{\varepsilon_1+\varepsilon_n}\cdot(\omega_2(Y_l^*)\otimes 1_{-s})$.

It follows from (3.10) that

$$X_{\varepsilon_1+\varepsilon_n}\cdot(\omega_2(Y_l^*)\otimes 1_{-s}) = X_{\varepsilon_1+\varepsilon_n}X_{-(\varepsilon_2+\varepsilon_n)}X_{-\varepsilon_1}\otimes 1_{-s} - X_{\varepsilon_1+\varepsilon_n}X_{-(\varepsilon_1+\varepsilon_n)}X_{-\varepsilon_2}\otimes 1_{-s}.$$

We observe from the second term. By the standard computation, we have

$$T_2 = -X_{\varepsilon_1+\varepsilon_n}X_{-(\varepsilon_1+\varepsilon_n)}X_{-\varepsilon_2}\otimes 1_{-s} = -H_{\varepsilon_1+\varepsilon_n}X_{-\varepsilon_2}\otimes 1_{-s} = s\lambda_{n-1}(H_{\varepsilon_1+\varepsilon_n})X_{-\varepsilon_2}\otimes 1_{-s}.$$

Observe that since $\lambda_{n-1} = \sum_{j=1}^{n-1}\varepsilon_j$, we have $\lambda_{n-1}(H_{\varepsilon_1+\varepsilon_n}) = 1$. Thus,

$$T_2 = s\lambda_{n-1}(H_{\varepsilon_1+\varepsilon_n})X_{-\varepsilon_2}\otimes 1_{-s} = sX_{-\varepsilon_2}\otimes 1_{-s}.$$

Similarly the standard computation shows that the first term amounts to

$$T_1 = X_{\varepsilon_1+\varepsilon_n}X_{-(\varepsilon_2+\varepsilon_n)}X_{-\varepsilon_1}\otimes 1_{-s} = -X_{-\varepsilon_2}\otimes 1_{-s}.$$

Therefore,

$$X_{\varepsilon_1+\varepsilon_n}\cdot(\omega_2(Y_l^*)\otimes 1_{-s}) = T_1+T_2 = (s-1)X_{-\varepsilon_2}\otimes 1_{-s}.$$

Now the assertion follows from Proposition 2.23. $\square$

Remark 3.12. Theorem 7.16 in [14] shows that the special values s_2 for the Ω_2 systems associated to type 1a special constituents $V(\mu+\epsilon)$ are given by $s_2 = \big(|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|/2\big)-1$, where $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|$ is the number of elements in $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))$. Now since $\Delta_{\mu+\epsilon_{n\gamma}}(\mathfrak{g}(1)) = \Delta_{\alpha_n+\gamma}(\mathfrak{g}(1)) = \{\varepsilon_1+\varepsilon_n, \varepsilon_2+\varepsilon_n\}\cup\{\varepsilon_1,\varepsilon_2\}$, the special value s_2 may be expressed as

$$s_2 = 1 = \frac{|\Delta_{\mu+\epsilon_{n\gamma}}(\mathfrak{g}(1))|}{2} - 1.$$

Thus the special value s_2 for the Ω_2 systems associated to type 1a and type 1b special constituents can be given in the same formula.

3.3. **The standardness of the map φ_{Ω_2}.** In the rest of this section we determine whether or not the map φ_{Ω_2} coming from the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system is standard.

Observe that, as $V(\mu+\epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n})$ and as the highest weights for $\mathfrak{l}_{n\gamma}$ and $\mathfrak{z}(\mathfrak{n})$ are ε_n and $-\varepsilon_{n-2}-\varepsilon_{n-1}$, respectively, we have

$$V(\mu+\epsilon_{n\gamma})^* = \mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\bar{\mathfrak{n}}) = V(-\varepsilon_{n-2}-\varepsilon_{n-1}+\varepsilon_n).$$

By Theorem 3.11, the special value s_2 for the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system is $s_2 = 1$. Therefore, by (2.10), the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system yields a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism

$$\varphi_{\Omega_2} : M_{\mathfrak{q}}((-\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n) - \lambda_{n-1} + \rho) \to M_{\mathfrak{q}}(-\lambda_{n-1} + \rho). \tag{3.13}$$

Theorem 3.14. *If $\mathfrak{q}$ is the maximal parabolic subalgebra of type $B_n(n-1)$ then the standard map φ_{std} between the generalized Verma modules in (3.13) is zero. Consequently, the map φ_{Ω_2} is non-standard.*

Proof. To prove this theorem, by Proposition 2.12, it suffices to show that there exits $\alpha_\nu \in \Pi(\mathfrak{l})$ so that $-\alpha_\nu - \lambda_{n-1} + \rho$ is linked to $(-\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n) - \lambda_{n-1} + \rho$. Observe that

$$-\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n = -2(\varepsilon_{n-1} - \varepsilon_n) - (\varepsilon_{n-2} - \varepsilon_{n-1}) - \varepsilon_n$$

with $\varepsilon_{n-1} - \varepsilon_n \in \Delta(\mathfrak{g}(1))$ and $\varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_n \in \Pi(\mathfrak{l})$. (See Appendix B.) We claim that $(\varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_{n-1} - \varepsilon_n)$ links $-\varepsilon_n - \lambda_{n-1} + \rho$ to $(-\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n) - \lambda_{n-1} + \rho$, that is,

$$s_{\varepsilon_{n-1}-\varepsilon_n} s_{\varepsilon_{n-2}-\varepsilon_{n-1}}(-\varepsilon_n - \lambda_{n-1} + \rho) = (-\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n) - \lambda_{n-1} + \rho$$

with

$$\langle -\varepsilon_n - \lambda_{n-1} + \rho, (\varepsilon_{n-2} - \varepsilon_{n-1})^\vee \rangle \in \mathbb{Z}_{\geq 0}$$

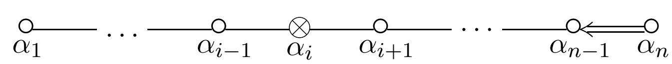
and

$$\langle s_{\varepsilon_{n-2}-\varepsilon_{n-1}}(-\varepsilon_n - \lambda_{n-1} + \rho), (\varepsilon_{n-1} - \varepsilon_n)^\vee \rangle \in \mathbb{Z}_{\geq 0}.$$

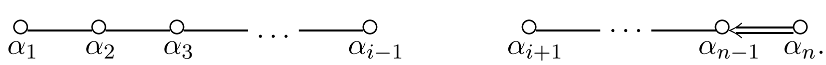
One can easily verify that these indeed hold. Now the theorem follows. □

4. Type 3 special constituent

In this section we study the Ω_2 system associated to the type 3 special constituent. It follows from Table 1 in the introduction that the type 3 special constituent occurs only when the parabolic subalgebra $\mathfrak{q}$ is of type $C_n(i)$ for $2 \leq i \leq n-1$. The simple root $\alpha_{\mathfrak{q}}$ that determines the parabolic subalgebra $\mathfrak{q}$ is then $\alpha_{\mathfrak{q}} = \alpha_i$. We write $\lambda_{\mathfrak{q}} = \lambda_i$ for the fundamental weight $\lambda_{\mathfrak{q}}$ for $\alpha_{\mathfrak{q}}$. The deleted Dynkin diagram for $\mathfrak{q}$ is



with connected subgraphs



Since α_1 is the unique simple root that is not orthogonal to the highest root γ, it follows from the subgraphs that $\mathfrak{l}_\gamma \cong \mathfrak{sl}(i, \mathbb{C})$ and $\mathfrak{l}_{n\gamma} \cong \mathfrak{sp}(n-i, \mathbb{C})$. Recall from Table 1 that the type 3 special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ is the irreducible constituent $V(\mu + \epsilon_\gamma) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$.

As in Section 3, our first goal is to show the L-intertwining operator $\tilde{\tau}_2|_{V(\mu+\epsilon_\gamma)^*}$ is not identically zero. To do so we again fix convenient root vectors for $\mathfrak{g}$. Observe that, as $\mathfrak{q}$ is of type $C_n(i)$, the Lie

TABLE 5. The values of $N_{\alpha,\beta}$ for roots α and β for $\mathfrak{sp}(2n,\mathbb{C})$ with indices $i<j<k$ when $\alpha+\beta$ is a positive root

Formula	α	β	$N_{\alpha,\beta}$
(1)	$\varepsilon_i+\varepsilon_k$	$\varepsilon_j-\varepsilon_k$	-1
(2)	$\varepsilon_i-\varepsilon_k$	$\varepsilon_j+\varepsilon_k$	$+1$
(3)	$\varepsilon_i+\varepsilon_k$	$-\varepsilon_j-\varepsilon_k$	$+1$
(4)	$\varepsilon_i-\varepsilon_k$	$-\varepsilon_j+\varepsilon_k$	$+1$
(5)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_j+\varepsilon_k$	-1
(6)	$\varepsilon_i-\varepsilon_j$	$\varepsilon_j+\varepsilon_k$	$+1$
(7)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_j-\varepsilon_k$	$+1$
(8)	$\varepsilon_i-\varepsilon_j$	$\varepsilon_j-\varepsilon_k$	$+1$
(9)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_i+\varepsilon_k$	-1
(10)	$-\varepsilon_i+\varepsilon_j$	$\varepsilon_i+\varepsilon_k$	$+1$
(11)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_i-\varepsilon_k$	$+1$
(12)	$-\varepsilon_i+\varepsilon_j$	$\varepsilon_i-\varepsilon_k$	$+1$
(13)	$\varepsilon_i+\varepsilon_j$	$\varepsilon_i-\varepsilon_j$	-2
(14)	$\varepsilon_i+\varepsilon_j$	$-\varepsilon_i+\varepsilon_j$	-2
(15)	$2\varepsilon_j$	$-\varepsilon_j+\varepsilon_k$	-1
(16)	$2\varepsilon_j$	$-\varepsilon_j-\varepsilon_k$	$+1$
(17)	$2\varepsilon_k$	$\varepsilon_j-\varepsilon_k$	-1
(18)	$-2\varepsilon_k$	$\varepsilon_j+\varepsilon_k$	$+1$

algebra $\mathfrak{g}$ under consideration is $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{C})$. For $1\leq j\leq n$, we write $\hat{j}=j+n$. If E_{ab} denotes the matrix with 1 in the (a,b) entry and 0 elsewhere then we take $\mathfrak{h}$ to be the set of diagonal matrices

$$H(h_1,\ldots,h_n)=h_1(E_{11}-E_{\hat{1}\hat{1}})+\cdots+h_n(E_{nn}-E_{\hat{n}\hat{n}})$$

with $h_j\in\mathbb{C}$. The positive roots are $\Delta^+=\{\varepsilon_j\pm\varepsilon_k\mid 1\leq j<k\leq n\}\cup\{2\varepsilon_j\mid 1\leq j\leq n\}$ with $\varepsilon_j(h_1,\ldots,h_n)=h_j$. We take the root vectors X_α as follows:

$$\begin{aligned}
X_{\varepsilon_j-\varepsilon_k}&=E_{jk}-E_{\hat{k}\hat{j}},\\
X_{\varepsilon_j+\varepsilon_k}&=E_{j\hat{k}}+E_{k\hat{j}},\\
X_{-(\varepsilon_j+\varepsilon_k)}&=E_{\hat{j}k}+E_{\hat{k}j},\\
X_{2\varepsilon_j}&=E_{j\hat{j}},\\
X_{-2\varepsilon_j}&=E_{\hat{j}j}.
\end{aligned}$$

For $\alpha,\beta\in\Delta$ with $\alpha+\beta\neq 0$, we again denote by $N_{\alpha,\beta}$ the constant so that $[X_\alpha,X_\beta]=N_{\alpha,\beta}X_{\alpha+\beta}$. Table 5 summarizes the values of $N_{\alpha,\beta}$ for $\alpha+\beta$ a positive root.

For $\alpha\in\Delta^+$ we set $H_\alpha=[X_\alpha,X_{-\alpha}]$. Namely,

$$H_{\varepsilon_j\pm\varepsilon_k}=(E_{jj}-E_{\hat{j}\hat{j}})\pm(E_{kk}-E_{\hat{k}\hat{k}})\quad\text{and}\quad H_{2\varepsilon_j}=E_{jj}-E_{\hat{j}\hat{j}}.$$

Now observe that if $T(X,Y):=\mathrm{Tr}(XY)$ then, as $T(\cdot,\cdot)|_{\mathfrak{h}_0\times\mathfrak{h}_0}$ is an inner product on a real form $\mathfrak{h}_0$ of $\mathfrak{h}$, there exists a positive constant c_o so that $\kappa(X,Y)=c_oT(X,Y)$. Since $T(X_\alpha,X_{-\alpha})$ takes the value of one for α long and two for α short, we have

$$\kappa(X_\alpha,X_{-\alpha})=\begin{cases}c_o & \text{if } \alpha \text{ is long}\\ 2c_o & \text{if } \alpha \text{ is short.}\end{cases}\tag{4.1}$$

Thus the dual vector X_α^* for X_α with respect to the Killing form is given by

$$X_\alpha^* = \begin{cases} (1/c_o)X_{-\alpha} & \text{if } \alpha \text{ is long} \\ (1/(2c_o))X_{-\alpha} & \text{if } \alpha \text{ is short.} \end{cases} \tag{4.2}$$

Now if $\Delta(\mathfrak{z}(\mathfrak{n}))_{\text{long}}$ (reps. $\Delta(\mathfrak{z}(\mathfrak{n}))_{\text{short}}$) is the set of long roots (reps. short roots) in $\Delta(\mathfrak{z}(\mathfrak{n}))$ then the element ω in (2.17) is given by

$$\begin{aligned} \omega &= \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{\gamma_j}^* \otimes X_{\gamma_j} \\ &= \frac{1}{c_o} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))_{\text{long}}} X_{-\gamma_j} \otimes X_{\gamma_j} + \frac{1}{2c_o} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))\text{short}} X_{-\gamma_j} \otimes X_{\gamma_j}. \end{aligned} \tag{4.3}$$

Observe that $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \leq j < k \leq i\} \cup \{2\varepsilon_j \mid 1 \leq j \leq i\}$. Thus,

$$\Delta(\mathfrak{z}(\mathfrak{n}))_{\text{long}} = \{2\varepsilon_j \mid 1 \leq j \leq i\} \quad \text{and} \quad \Delta(\mathfrak{z}(\mathfrak{n}))_{\text{short}} = \{\varepsilon_j + \varepsilon_k \mid 1 \leq j < k \leq i\}.$$

Therefore, (4.3) reads

$$\omega = \frac{1}{c_o} \sum_{j=1}^{i} X_{-2\varepsilon_j} \otimes X_{2\varepsilon_j} + \frac{1}{2c_o} \sum_{1 \leq j < k \leq i} X_{-(\varepsilon_j + \varepsilon_k)} \otimes X_{\varepsilon_j + \varepsilon_k}.$$

Thus the τ_2 map in (2.18) may be expressed as

$$\begin{aligned} \tau_2(X) &= \frac{1}{2}\big(\mathrm{ad}(X)^2 \otimes \mathrm{Id}\big)\omega \\ &= \frac{1}{2c_o} \sum_{j=1}^{i} \mathrm{ad}(X)^2 X_{-2\varepsilon_j} \otimes X_{2\varepsilon_j} + \frac{1}{4c_o} \sum_{1 \leq j < k \leq i} \mathrm{ad}(X)^2 X_{-(\varepsilon_j + \varepsilon_k)} \otimes X_{\varepsilon_j + \varepsilon_k}. \end{aligned} \tag{4.4}$$

As for the case that $\mathfrak{q}$ is of type $B_n(n-1)$, Proposition 7.3 of [14] showed that the τ_2 map is not identically zero.

4.1. **Lowest weight vector for $V(\mu + \epsilon_\gamma)^*$.** As for the case of type 1b special constituent it is necessary to show that the linear map $\tilde{\tau}_2|_{V(\mu+\epsilon_\gamma)^*} : V(\mu + \epsilon_\gamma)^* \to \mathcal{P}^2(\mathfrak{g}(1))$ is not identically zero. We will again achieve it by showing that $\tilde{\tau}_2(Y_l^*)(X)$ is a non-zero polynomial on $\mathfrak{g}(1)$, where Y_l^* is a lowest weight vector for $V(\mu + \epsilon_\gamma)^*$.

When a special constituent is of type 1b, as $V(\mu + \epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$, it was easy to find a lowest weight vector for $V(\mu + \epsilon_{n\gamma})^*$. In contrast, in the present case, since $V(\mu + \epsilon_\gamma) \subsetneq \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$, we cannot use the same idea. So our first goal is to find an explicit form of a lowest weight vector for $V(\mu + \epsilon_\gamma)^*$. To do so we now observe a highest weight vector for $V(\mu + \epsilon_\gamma)$.

If $\mathrm{pr}_{\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})} : \mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) \to \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ is the projection map from $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ onto $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ then we claim that $\mathrm{pr}_{\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu + X_{\epsilon_\gamma}))$ is a highest weight vector for $V(\mu + \epsilon_\gamma)$. The following two technical lemmas will simplify the expositions of the proof.

Lemma 4.5. *For $Z \in \mathfrak{l}$ and $X_1, X_2 \in \mathfrak{g}(1)$, we have*

$$Z \cdot \big(\mathrm{ad}(X_1)\mathrm{ad}(X_2) \otimes \mathrm{Id}\big)\omega = \big((\mathrm{ad}([Z, X_1])\mathrm{ad}(X_2) + \mathrm{ad}(X_1)\mathrm{ad}([Z, X_2])) \otimes \mathrm{Id}\big)\omega, \tag{4.6}$$

where dot $(\cdot)$ denotes the usual Lie algebra action on tensor products.

Proof. As this lemma simply follows from the arguments used in the proof for Proposition 7.5 of [14], we omit the proof. $\square$

Lemma 4.7. *We have*

$$\begin{aligned}&\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu+X_{\epsilon_\gamma}))\\&=\frac{1}{2}\big(\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_\mu)\mathrm{ad}(X_{\epsilon_\gamma})\otimes\mathrm{Id})\omega)+\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}(X_\mu)\otimes\mathrm{Id})\omega)\big).\end{aligned}$$

Proof. Since

$$\begin{aligned}\tau_2(X_\mu+X_{\epsilon_\gamma})&=\frac{1}{2}\big(\mathrm{ad}(X_\mu+X_{\epsilon_\gamma})^2\otimes\mathrm{Id}\big)\omega\\&=\frac{1}{2}\big((\mathrm{ad}(X_\mu)^2\otimes\mathrm{Id})\omega+(\mathrm{ad}(X_\mu)\mathrm{ad}(X_{\epsilon_\gamma})\otimes\mathrm{Id})\omega\\&\qquad+(\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}(X_\mu)\otimes\mathrm{Id})\omega+(\mathrm{ad}(X_{\epsilon_\gamma})^2\otimes\mathrm{Id})\omega\big),\end{aligned}\tag{4.8}$$

we have

$$\begin{aligned}\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu+X_{\epsilon_\gamma}))&=\frac{1}{2}\big(\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_\mu)^2\otimes\mathrm{Id})\omega)+\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_\mu)\mathrm{ad}(X_{\epsilon_\gamma})\otimes\mathrm{Id})\omega)\\&\quad+\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}(X_\mu)\otimes\mathrm{Id})\omega)+\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_{\epsilon_\gamma})^2\otimes\mathrm{Id})\omega)\big).\end{aligned}\tag{4.9}$$

Observe that $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_\mu)^2\otimes\mathrm{Id})\omega)=0$. Indeed, as $\mu=\varepsilon_1+\varepsilon_{i+1}$ and $\omega=(1/c_o)\sum_{j=1}^{i}X_{-2\varepsilon_j}\otimes X_{2\varepsilon_j}+(1/2c_o)\sum_{1\leq j<k\leq i}X_{-(\varepsilon_j+\varepsilon_k)}\otimes X_{\varepsilon_j+\varepsilon_k}$ (see (4.3)), we have

$$\begin{aligned}&(\mathrm{ad}(X_\mu)^2\otimes\mathrm{Id})\omega\\&=\frac{1}{c_o}\sum_{j=1}^{i}\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})^2X_{-2\varepsilon_j}\otimes X_{2\varepsilon_j}+\frac{1}{2c_o}\sum_{1\leq j<k\leq i}\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})^2X_{-(\varepsilon_j+\varepsilon_k)}\otimes X_{\varepsilon_j+\varepsilon_k}\\&=\frac{1}{c_o}\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})^2X_{-2\varepsilon_1}\otimes X_{2\varepsilon_1}\\&=\frac{1}{c_o}N_{\varepsilon_1+\varepsilon_{i+1},-2\varepsilon_1}N_{\varepsilon_1+\varepsilon_{i+1},-\varepsilon_1+\varepsilon_{i+1}}X_{2\varepsilon_{i+1}}\otimes X_{2\varepsilon_1}.\end{aligned}$$

Clearly, $X_{2\varepsilon_{i+1}}\otimes X_{2\varepsilon_1}\in\mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n})$. Thus, $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_\mu)^2\otimes\mathrm{Id})\omega)=0$. It can be shown similarly that $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_{\epsilon_\gamma})^2\otimes\mathrm{Id})\omega)=0$. Now the proposed equality follows from (4.9). $\square$

Proposition 4.10. *The vector* $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu+X_{\epsilon_\gamma}))$ *is a highest weight vector for* $V(\mu+\epsilon_\gamma)$.

Proof. We start with showing that $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu+X_{\epsilon_\gamma}))$ has weight $\mu+\epsilon_\gamma$. Since the $\mathfrak{l}$-action commutes with the projection map $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}$, by Lemma 4.7, for $H\in\mathfrak{h}\subset\mathfrak{l}$, we have

$$\begin{aligned}&H\cdot\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu+X_{\epsilon_\gamma}))\\&=\frac{1}{2}\big(\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(H\cdot\mathrm{ad}(X_\mu)\mathrm{ad}(X_{\epsilon_\gamma})\otimes\mathrm{Id})\omega)+\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(H\cdot\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}(X_\mu)\otimes\mathrm{Id})\omega)\big)\\&=\frac{1}{2}\big(\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}\big((\mathrm{ad}([H,X_\mu])\mathrm{ad}(X_{\epsilon_\gamma})+\mathrm{ad}(X_\mu)\mathrm{ad}([H,X_{\epsilon_\gamma}])\big)\otimes\mathrm{Id})\omega)\\&\qquad+\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}\big((\mathrm{ad}([H,X_{\epsilon_\gamma}])\mathrm{ad}(X_\mu)+\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}([H,X_\mu]))\otimes\mathrm{Id})\omega)\big)\\&=\frac{(\mu+\epsilon_\gamma)(H)}{2}\big(\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_\mu)\mathrm{ad}(X_{\epsilon_\gamma})\otimes\mathrm{Id})\omega)+\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}(X_\mu)\otimes\mathrm{Id})\omega)\big)\\&=(\mu+\epsilon_\gamma)(H)\ \mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu+X_{\epsilon_\gamma})).\end{aligned}$$

Note that Lemma 4.5 is applied from line two to line three.

Next we show that $\mathrm{pr}_{\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})}(\tau_2(X_\mu + X_{\epsilon_\gamma}))$ is a highest weight vector. Let $\alpha \in \Pi(\mathfrak{l})$. As the $\mathfrak{l}$-action commutes with $\mathrm{pr}_{\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})}$, we first observe $X_\alpha \cdot \tau_2(X_\mu + X_{\epsilon_\gamma})$. It follows from (4.8) that $X_\alpha \cdot \tau_2(X_\mu + X_{\epsilon_\gamma})$ is given by

$$\begin{aligned}
&X_\alpha \cdot \tau_2(X_\mu + X_{\epsilon_\gamma})\\
&= \frac{1}{2}\big(X_\alpha \cdot (\mathrm{ad}(X_\mu)^2 \otimes \mathrm{Id})\omega + X_\alpha \cdot (\mathrm{ad}(X_\mu)\mathrm{ad}(X_{\epsilon_\gamma}) \otimes \mathrm{Id})\omega\\
&\qquad + X_\alpha \cdot (\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}(X_\mu) \otimes \mathrm{Id})\omega + X_\alpha \cdot (\mathrm{ad}(X_{\epsilon_\gamma})^2 \otimes \mathrm{Id})\omega\big).
\end{aligned}$$

If $Z = X_\alpha$ in (4.6) then, as $[X_\alpha, X_\mu] = 0$, we obtain

$$\begin{aligned}
&X_\alpha \cdot \tau_2(X_\mu + X_{\epsilon_\gamma})\\
&= \frac{1}{2}\big(\mathrm{ad}(X_\mu)\mathrm{ad}([X_\alpha, X_{\epsilon_\gamma}]) \otimes \mathrm{Id}\big)\omega + \big(\mathrm{ad}([X_\alpha, X_{\epsilon_\gamma}])\mathrm{ad}(X_\mu) \otimes \mathrm{Id}\big)\omega\\
&\quad + \big(\mathrm{ad}([X_\alpha, X_{\epsilon_\gamma}])\mathrm{ad}(X_{\epsilon_\gamma}) \otimes \mathrm{Id}\big)\omega + \big(\mathrm{ad}(X_{\epsilon_\gamma})\mathrm{ad}([X_\alpha, X_{\epsilon_\gamma}]) \otimes \mathrm{Id}\big)\omega.
\end{aligned} \tag{4.11}$$

Recall from Tables 2 and 4 in Section 6 of [14] that we have $\mu + \epsilon_\gamma = \varepsilon_1 + \varepsilon_2$ with $\mu = \varepsilon_1 + \varepsilon_{i+1}$ and $\epsilon_\gamma = \varepsilon_2 - \varepsilon_{i+1}$. Since $\Pi(\mathfrak{l}) = \{\varepsilon_j - \varepsilon_{j+1} \ : \ 1 \leq j \leq n-1 \text{ with } j \neq i\} \cup \{2\varepsilon_n\}$, it follows that $\alpha + \epsilon_\gamma \in \Delta$ if and only if $\alpha = \varepsilon_1 - \varepsilon_2$. So it suffices to consider the case that $\alpha = \varepsilon_1 - \varepsilon_2$. As $(\varepsilon_1 - \varepsilon_2) + 2(\varepsilon_2 - \varepsilon_{i+1}) \notin \Delta$, we have $\mathrm{ad}(X_{\varepsilon_1-\varepsilon_2})\mathrm{ad}(X_{\varepsilon_2-\varepsilon_{i+1}}) = \mathrm{ad}(X_{\varepsilon_2-\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_2})$. Therefore, if $\alpha = \varepsilon_1 - \varepsilon_2$ then, as $\mu = \varepsilon_1 + \varepsilon_{i+1}$ and $\epsilon_\gamma + \alpha = \varepsilon_1 - \varepsilon_{i+1}$, (4.11) becomes

$$\begin{aligned}
&X_{\varepsilon_1-\varepsilon_2} \cdot \tau_2(X_{\varepsilon_1+\varepsilon_{i+1}} + X_{\varepsilon_2-\varepsilon_{i+1}})\\
&= \frac{N_{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_{i+1}}}{2}\Big(\big(\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega + \big(\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega\\
&\qquad\qquad + \big(2\mathrm{ad}(X_{\varepsilon_2-\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega\Big).
\end{aligned}$$

Table 5 shows that $N_{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_{i+1}} = 1$. Therefore,

$$\begin{aligned}
&X_{\varepsilon_1-\varepsilon_2} \cdot \tau_2(X_{\varepsilon_1+\varepsilon_{i+1}} + X_{\varepsilon_2-\varepsilon_{i+1}})\\
&= \frac{1}{2}\Big(\big(\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega + \big(\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega\\
&\qquad\qquad + \big(2\mathrm{ad}(X_{\varepsilon_2-\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega\Big).
\end{aligned}$$

Now we consider the contribution from each term separately. A direct computation shows that the first term is given by

$$\begin{aligned}
T_1 &= \big(\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega\\
&= \frac{1}{c_o}\sum_{j=1}^{i} \mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}})X_{-2\varepsilon_j} \otimes X_{2\varepsilon_j}\\
&\qquad\qquad + \frac{1}{2c_o}\sum_{1\leq j<k\leq i} \mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}})X_{-(\varepsilon_j+\varepsilon_k)} \otimes X_{\varepsilon_j+\varepsilon_k}\\
&= \frac{1}{c_o}N_{\varepsilon_1-\varepsilon_{i+1},-2\varepsilon_1}H_{\varepsilon_1+\varepsilon_{i+1}} \otimes X_{2\varepsilon_1}\\
&\qquad\qquad + \frac{1}{2c_o}\sum_{k=2}^{i} N_{\varepsilon_1-\varepsilon_{i+1},-(\varepsilon_1+\varepsilon_k)}N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_k+\varepsilon_{i+1})}X_{\varepsilon_1-\varepsilon_k} \otimes X_{\varepsilon_1+\varepsilon_k}\\
&= \frac{-1}{c_o}H_{\varepsilon_1+\varepsilon_{i+1}} \otimes X_{2\varepsilon_1} - \frac{1}{2c_o}\sum_{k=2}^{i} X_{\varepsilon_1-\varepsilon_k} \otimes X_{\varepsilon_1+\varepsilon_k}.
\end{aligned}$$

Similarly we have

$$T_2 = \big(\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1+\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega = \frac{1}{c_o}H_{\varepsilon_1-\varepsilon_{i+1}} \otimes X_{2\varepsilon_1} + \frac{1}{2c_o}\sum_{k=2}^{i} X_{\varepsilon_1-\varepsilon_k} \otimes X_{\varepsilon_1+\varepsilon_k}$$

and

$$T_3 = \big(2\mathrm{ad}(X_{\varepsilon_2-\varepsilon_{i+1}})\mathrm{ad}(X_{\varepsilon_1-\varepsilon_{i+1}}) \otimes \mathrm{Id}\big)\omega = \frac{4}{c_o}X_{2\varepsilon_{i+1}} \otimes X_{-(\varepsilon_1+\varepsilon_2)}.$$

Therefore,

$$\begin{aligned}
&X_{\varepsilon_1-\varepsilon_2} \cdot \tau_2(X_{\varepsilon_1+\varepsilon_{i+1}} + X_{\varepsilon_2-\varepsilon_{i+1}})\\
&= T_1 + T_2 + T_3\\
&= \frac{-1}{c_o}H_{\varepsilon_1+\varepsilon_{i+1}} \otimes X_{2\varepsilon_1} - \frac{1}{2c_o}\sum_{k=2}^{i} X_{\varepsilon_1-\varepsilon_k} \otimes X_{\varepsilon_1+\varepsilon_k}\\
&\quad + \frac{1}{c_o}H_{\varepsilon_1-\varepsilon_{i+1}} \otimes X_{2\varepsilon_1} + \frac{1}{2c_o}\sum_{k=2}^{i} X_{\varepsilon_1-\varepsilon_k} \otimes X_{\varepsilon_1+\varepsilon_k} + \frac{4}{c_o}X_{2\varepsilon_{i+1}} \otimes X_{-(\varepsilon_1+\varepsilon_2)}\\
&= \frac{1}{c_o}(H_{\varepsilon_1-\varepsilon_{i+1}} - H_{\varepsilon_1+\varepsilon_{i+1}}) \otimes X_{2\varepsilon_1} + \frac{4}{c_o}X_{2\varepsilon_{i+1}} \otimes X_{-(\varepsilon_1+\varepsilon_2)}.
\end{aligned} \tag{4.12}$$

Observe that $\mathfrak{h} \cap \mathfrak{l}_\gamma$ is spanned by the elements $H_{\varepsilon_j-\varepsilon_{j+1}} = (E_{jj} - E_{\hat{j},\hat{j}}) - (E_{j+1,j+1} - E_{\widehat{j+1},\widehat{j+1}})$ for $1 \leq j \leq i-1$. Since $H_{\varepsilon_1-\varepsilon_{i+1}} - H_{\varepsilon_1+\varepsilon_{i+1}} = -2(E_{i+1,i+1} - E_{\widehat{i+1},\widehat{i+1}})$, it follows that $H_{\varepsilon_1-\varepsilon_{i+1}} - H_{\varepsilon_1+\varepsilon_{i+1}} \notin \mathfrak{h} \cap \mathfrak{l}_\gamma$. As $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(X_{2\varepsilon_{i+1}} \otimes X_{-(\varepsilon_1+\varepsilon_2)}) = 0$, we then obtain

$$\begin{aligned}
&X_{\varepsilon_1-\varepsilon_2} \cdot \mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(\tau_2(X_{\varepsilon_1+\varepsilon_{i+1}} + X_{\varepsilon_2-\varepsilon_{i+1}}))\\
&= \mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}(X_{\varepsilon_1-\varepsilon_2} \cdot \tau_2(X_{\varepsilon_1+\varepsilon_{i+1}} + X_{\varepsilon_2-\varepsilon_{i+1}}))\\
&= \frac{1}{c_o}\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}\big((H_{\varepsilon_1-\varepsilon_{i+1}} - H_{\varepsilon_1+\varepsilon_{i+1}}) \otimes X_{2\varepsilon_1}\big) + \frac{4}{c_o}\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})}\big(X_{2\varepsilon_{i+1}} \otimes X_{-(\varepsilon_1+\varepsilon_2)}\big)\\
&= 0.
\end{aligned}$$

$\square$

Now we define the "opposite" τ_2 map by

$$\begin{aligned}\bar{\tau}_2 : \mathfrak{g}(-1) &\to \mathfrak{g}(0) \otimes \mathfrak{g}(-2)\\ X^* &\mapsto \frac{1}{2}\,\big(\mathrm{ad}(X^*)^2 \otimes \mathrm{Id}\big)\bar{\omega}\end{aligned}$$

with

$$\begin{aligned}\bar{\omega} &= \sum_{\gamma_j \in \mathfrak{z}(\mathfrak{n})} X_\gamma \otimes X^*_\gamma\\ &= \frac{1}{c_o}\sum_{j=1}^{i} X_{2\varepsilon_j} \otimes X_{-2\varepsilon_j} + \frac{1}{2c_o}\sum_{1\leq j<k\leq i} X_{\varepsilon_j+\varepsilon_k} \otimes X_{-(\varepsilon_j+\varepsilon_k)}.\end{aligned}$$

It follows from the same arguments in the proof for Lemma 3.3 and Proposition 7.3 in [14] that the $\bar{\tau}_2$ map is not identically zero and L-equivariant. Let $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})} : \mathfrak{l}\otimes\mathfrak{z}(\bar{\mathfrak{n}}) \to \mathfrak{l}_\gamma \otimes \mathfrak{z}(\bar{\mathfrak{n}})$ be the projection map from $\mathfrak{l}\otimes\mathfrak{z}(\bar{\mathfrak{n}})$ onto $\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})$.

Proposition 4.13. *The vector* $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\bar{\tau}_2(X_{-\mu}+X_{-\epsilon_\gamma}))$ *is a lowest weight vector for* $V(\mu+\epsilon_\gamma)^*$.

Proof. This proposition immediately follows from the arguments used in the proof for Proposition 4.10, by replacing positive (resp. negative) roots with negative (resp. positive) roots. ☐

We set

$$Y_l^* := \frac{8c_o^2}{i+1}\,\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\bar{\tau}_2(X_{-\mu}+X_{-\epsilon_\gamma})). \tag{4.14}$$

It follows from Proposition 4.13 that Y_l^* is a lowest weight vector for $V(\mu+\epsilon_\gamma)^*$. In the next subsection we compute $\tilde{\tau}_2(Y_l^*)(X)$. To the end we give an explicit form for Y_l^*.

Lemma 4.15. *We have*

$$\begin{aligned}Y_l^* = {}&\frac{4c_o}{i+1}X_{\varepsilon_1-\varepsilon_2}\otimes X_{-2\varepsilon_1} - \frac{4c_o}{i+1}X_{-(\varepsilon_1-\varepsilon_2)}\otimes X_{-2\varepsilon_2}\\ &+\frac{2c_o}{i+1}\sum_{k=3}^{i}X_{-(\varepsilon_2-\varepsilon_k)}\otimes X_{-(\varepsilon_1+\varepsilon_k)} - \frac{2c_o}{i+1}\sum_{k=3}^{i}X_{-(\varepsilon_1-\varepsilon_k)}\otimes X_{-(\varepsilon_2+\varepsilon_k)}\\ &-\frac{2c_o}{i+1}H_{\varepsilon_1-\varepsilon_2}\otimes X_{-(\varepsilon_1+\varepsilon_2)}.\end{aligned}$$

Proof. By using the same arguments for Lemma 4.7, one can obtain

$$\begin{aligned}&\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\tau_2(X_{-\mu}+X_{-\epsilon_\gamma}))\\ &=\frac{1}{2}\big(\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\mathrm{ad}(X_{-\mu})\mathrm{ad}(X_{-\epsilon_\gamma})\otimes\mathrm{Id})\bar{\omega}) + \mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\mathrm{ad}(X_{-\epsilon_\gamma})\mathrm{ad}(X_{-\mu})\otimes\mathrm{Id})\bar{\omega})\big).\end{aligned} \tag{4.16}$$

A direct computation shows that

$$\begin{aligned}
&\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\mathrm{ad}(X_{-\mu})\mathrm{ad}(X_{-\epsilon_\gamma})\otimes \mathrm{Id})\bar{\omega})\\
&=\frac{1}{c_o}\sum_{j=1}^{i}\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}\big(\mathrm{ad}(X_{-(\varepsilon_1+\varepsilon_{i+1})})\mathrm{ad}(X_{-(\varepsilon_2-\varepsilon_{i+1})})X_{2\varepsilon_j}\otimes X_{-2\varepsilon_j}\big)\\
&\quad+\frac{1}{2c_o}\sum_{1\leq j<k\leq i}\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}\big(\mathrm{ad}(X_{-(\varepsilon_1+\varepsilon_{i+1})})\mathrm{ad}(X_{-(\varepsilon_2-\varepsilon_{i+1})})X_{\varepsilon_j+\varepsilon_k}\otimes X_{-(\varepsilon_j+\varepsilon_k)}\big)\\
&=-\frac{1}{c_o}X_{-(\varepsilon_1-\varepsilon_2)}\otimes X_{-2\varepsilon_2}-\frac{1}{2c_o}\,\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(H_{\varepsilon_1+\varepsilon_{i+1}}\otimes X_{-(\varepsilon_1+\varepsilon_2)})-\frac{1}{2c_o}\sum_{k=3}^{i}X_{-(\varepsilon_1-\varepsilon_k)}\otimes X_{-(\varepsilon_2+\varepsilon_k)}.
\end{aligned}$$

Observe that we have

$$H_{\varepsilon_1+\varepsilon_{i+1}}=\frac{1}{i}\sum_{j=1}^{i}(E_{jj}-E_{\hat{j}\hat{j}})+\frac{1}{i}\sum_{k=2}^{i}H_{\varepsilon_1-\varepsilon_k}+H_{2\varepsilon_{i+1}}.$$

Since $(1/i)\sum_{j=1}^{i}(E_{jj}-E_{\hat{j}\hat{j}})\in\mathfrak{z}(\mathfrak{l})$ and $H_{2\varepsilon_{i+1}}\in\mathfrak{h}\cap\mathfrak{l}_{n\gamma}$, it follows that

$$\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(H_{\varepsilon_1+\varepsilon_{i+1}}\otimes X_{-(\varepsilon_1+\varepsilon_2)})=\frac{1}{i}\sum_{k=2}^{i}H_{\varepsilon_1-\varepsilon_k}\otimes X_{-(\varepsilon_1+\varepsilon_2)}.$$

Therefore, $\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\mathrm{ad}(X_{-\mu})\mathrm{ad}(X_{-\epsilon_\gamma})\otimes\mathrm{Id})\bar{\omega})$ is given by

$$\begin{aligned}
&\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\mathrm{ad}(X_{-\mu})\mathrm{ad}(X_{-\epsilon_\gamma})\otimes \mathrm{Id})\bar{\omega})\\
&=-\frac{1}{c_o}X_{-(\varepsilon_1-\varepsilon_2)}\otimes X_{-2\varepsilon_2}-\frac{1}{2ic_o}\sum_{k=2}^{i}H_{\varepsilon_1-\varepsilon_k}\otimes X_{-(\varepsilon_1+\varepsilon_2)}-\frac{1}{2c_o}\sum_{k=3}^{i}X_{-(\varepsilon_1-\varepsilon_k)}\otimes X_{-(\varepsilon_2+\varepsilon_k)}. \quad (4.17)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\mathrm{pr}_{\mathfrak{l}_\gamma\otimes\mathfrak{z}(\bar{\mathfrak{n}})}(\mathrm{ad}(X_{-\epsilon_\gamma})\mathrm{ad}(X_{-\mu})\otimes \mathrm{Id})\bar{\omega}) \quad (4.18)\\
&=\frac{1}{c_o}X_{\varepsilon_1-\varepsilon_2}\otimes X_{-2\varepsilon_1}-\frac{1}{2ic_o}(H_{\varepsilon_1-\varepsilon_2}-\sum_{k=3}^{i}H_{\varepsilon_2-\varepsilon_k})\otimes X_{-(\varepsilon_1+\varepsilon_2)}+\frac{1}{2c_o}\sum_{k=3}^{i}X_{-(\varepsilon_2-\varepsilon_k)}\otimes X_{-(\varepsilon_1+\varepsilon_k)}.
\end{aligned}$$

By substituting (4.17) and (4.18) into (4.16) and multiplying the resulting equation by $8c_o^2/(i+1)$, one obtains

$$\begin{aligned}
Y_l^*&=\frac{4c_o}{i+1}X_{\varepsilon_1-\varepsilon_2}\otimes X_{-2\varepsilon_1}-\frac{4c_o}{i+1}X_{-(\varepsilon_1-\varepsilon_2)}\otimes X_{-2\varepsilon_2}\\
&\quad+\frac{2c_o}{i+1}\sum_{k=3}^{i}X_{-(\varepsilon_2-\varepsilon_k)}\otimes X_{-(\varepsilon_1+\varepsilon_k)}-\frac{2c_o}{i+1}\sum_{k=3}^{i}X_{-(\varepsilon_1-\varepsilon_k)}\otimes X_{-(\varepsilon_2+\varepsilon_k)}\\
&\quad-\frac{2c_o}{i(i+1)}(\sum_{k=2}^{i}H_{\varepsilon_1-\varepsilon_k}+H_{\varepsilon_1-\varepsilon_2}-\sum_{k=3}^{i}H_{\varepsilon_2-\varepsilon_k})\otimes X_{-(\varepsilon_1+\varepsilon_2)}.
\end{aligned}$$

Now the proposed equality follows from manipulating the elements in the Cartan subalgebra. $\square$

4.2. **The $\tilde{\tau}_2|_{V(\mu+\epsilon_\gamma)^*}$ map.** Now we show that the map $\tilde{\tau}_2|_{V(\mu+\epsilon_\gamma)^*}$ is not identically zero. To do so, we recall several essential ingredients. First observe that, as the duality is with respect to the Killing form κ, if $Y^* = X_\alpha \otimes X_{-\gamma_j} \in \mathfrak{l}_\gamma \otimes \mathfrak{z}(\bar{\mathfrak{n}})$ and $X_\beta \otimes X_{\gamma_k} \in \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ then $Y^*(X_\beta \otimes X_{\gamma_k})$ is given by $Y^*(X_\beta \otimes X_{\gamma_k}) = \kappa(X_\alpha, X_\beta)\kappa(X_{-\gamma_k}, X_{\gamma_j})$. As observed in (4.1), we have

$$\kappa(X_\alpha, X_{-\alpha}) = \begin{cases} c_o & \text{if } \alpha \text{ is long} \\ 2c_o & \text{if } \alpha \text{ is short.} \end{cases}$$

Finally we recall from (3.4) that if $W \subset \mathfrak{g}$ is an $\mathrm{ad}(\mathfrak{h})$-invariant subspace then, for any weight $\nu \in \mathfrak{h}^*$, we write $\Delta_\nu(W) = \{\alpha \in \Delta(W) \mid \nu - \alpha \in \Delta\}$.

Proposition 4.19. *The L-intertwining operator $\tilde{\tau}_2|_{V(\mu+\epsilon_\gamma)^*}$ is not identically zero.*

Proof. Take lowest weight vector Y_l^* as in (4.14). We show that $\tilde{\tau}_2(Y_l^*)(X)$ is a non-zero polynomial on $\mathfrak{g}(1)$. As $\tau_2(X) = (1/2)\sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{ad}(X)^2 X^*_{\gamma_j} \otimes X_{\gamma_j}$, by Lemma 4.15, the polynomial $\tilde{\tau}_2(Y_l^*)(X)$ may express as a sum of five terms. We consider the contribution from each term separately. We start with observing the contribution from the first term. We have

$$\begin{aligned} T_1 &= \frac{2c_o}{i+1} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \kappa(X_{\varepsilon_1-\varepsilon_2}, \mathrm{ad}(X)^2 X^*_{\gamma_j})\kappa(X_{-2\varepsilon_1}, X_{\gamma_j}) \\ &= \frac{2c_o}{i+1}\kappa(X_{\varepsilon_1-\varepsilon_2}, \mathrm{ad}(X)^2 X_{-2\varepsilon_1}). \end{aligned} \tag{4.20}$$

Write $X = \sum_{\alpha \in \Delta(\mathfrak{g}(1))} \eta_\alpha X_\alpha$, where $\eta_\alpha \in \mathfrak{n}^*$ is the coordinate dual to X_α with respect to the Killing form. Then,

$$\begin{aligned} (4.20) &= \frac{2c_o}{i+1}\kappa(X_{\varepsilon_1-\varepsilon_2}, \mathrm{ad}(X)^2 X_{-2\varepsilon_1}) \\ &= \frac{2c_o}{i+1} \sum_{\alpha,\beta \in \Delta(\mathfrak{g}(1))} \eta_\alpha \eta_\beta \kappa(X_{\varepsilon_1-\varepsilon_2}, [X_\beta, [X_\alpha, X_{-2\varepsilon_1}]]) \\ &= \frac{2c_o}{i+1} \sum_{\alpha,\beta \in \Delta(\mathfrak{g}(1))} \eta_\alpha \eta_\beta \kappa([X_{\varepsilon_1-\varepsilon_2}, X_\beta], [X_\alpha, X_{-2\varepsilon_1}]) \\ &= \frac{2c_o}{i+1} \sum_{\substack{\alpha \in \Delta_{2\varepsilon_1}(\mathfrak{g}(1)) \\ \beta \in \Delta^{\varepsilon_1-\varepsilon_2}(\mathfrak{g}(1))}} N_{\varepsilon_1-\varepsilon_2,\beta} N_{\alpha,-2\varepsilon_1} \eta_\alpha \eta_\beta \kappa(X_{\beta+(\varepsilon_1-\varepsilon_2)}, X_{\alpha-2\varepsilon_1}), \end{aligned} \tag{4.21}$$

where $\Delta^{\varepsilon_1-\varepsilon_2}(\mathfrak{g}(1)) = \{\alpha \in \Delta(\mathfrak{g}(1)) \mid (\varepsilon_1 - \varepsilon_2) + \alpha \in \Delta\}$. Observe that $\kappa(X_{\beta+(\varepsilon_1-\varepsilon_2)}, X_{\alpha-2\varepsilon_1}) \neq 0$ if and only if $\alpha \in \Delta_{2\varepsilon_1}(\mathfrak{g}(1))$. Indeed, first one may see from (4.21) that $\kappa(X_{\beta+(\varepsilon_1-\varepsilon_2)}, X_{\alpha-2\varepsilon_1}) \neq 0$ if and only if $\beta = (\varepsilon_1 + \varepsilon_2) - \alpha$; equivalently, the value of the Killing form is non-zero if and only if $\alpha \in \Delta_{2\varepsilon_1}(\mathfrak{g}(1)) \cap \Delta_{\varepsilon_1+\varepsilon_2}(\mathfrak{g}(1))$. By inspection, we have $\Delta_{2\varepsilon_1}(\mathfrak{g}(1)) = \{\varepsilon_1 \pm \varepsilon_j \mid i+1 \leq j \leq n\}$. Thus, for any $\alpha \in \Delta_{2\varepsilon_1}(\mathfrak{g}(1))$, it follows that $(\varepsilon_1 + \varepsilon_2) - \alpha \in \Delta$. Therefore $\Delta_{2\varepsilon_1}(\mathfrak{g}(1)) \cap \Delta_{\varepsilon_1+\varepsilon_2}(\mathfrak{g}(1)) = \Delta_{2\varepsilon_1}(\mathfrak{g}(1))$. Now, if (α, β) denotes a pair so that $\kappa(X_{\beta+(\varepsilon_1-\varepsilon_2)}, X_{\alpha-2\varepsilon_1}) \neq 0$ then $(\alpha, \beta) = (\varepsilon_1 \pm$

$\varepsilon_j, \varepsilon_2 \mp \varepsilon_j)$ for $i+1 \leq j \leq n$ with respect to the signs. Therefore,

$$\begin{aligned}
(4.21) &= \frac{2c_o}{i+1} \sum_{\substack{\alpha \in \Delta_{2\varepsilon_1}(\mathfrak{g}(1)) \\ \beta \in \Delta^{\varepsilon_1-\varepsilon_2}(\mathfrak{g}(1))}} N_{\varepsilon_1-\varepsilon_2,\beta} N_{\alpha,-2\varepsilon_1} \eta_\alpha \eta_\beta \kappa(X_{\beta+(\varepsilon_1-\varepsilon_2)}, X_{\alpha-2\varepsilon_1}) \\
&= \frac{2c_o}{i+1} \sum_{\alpha \in \Delta_{2\varepsilon_1}(\mathfrak{g}(1))} N_{\varepsilon_1-\varepsilon_2,(\varepsilon_1+\varepsilon_2)-\alpha} N_{\alpha,-2\varepsilon_1} \eta_\alpha \eta_{(\varepsilon_1+\varepsilon_2)-\alpha} \kappa(X_{2\varepsilon_1-\alpha}, X_{\alpha-2\varepsilon_1}) \\
&= \frac{2c_o}{i+1} \sum_{j=i+1}^{n} N_{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_j} N_{\varepsilon_1+\varepsilon_j,-2\varepsilon_1} \eta_{\varepsilon_1+\varepsilon_j} \eta_{\varepsilon_2-\varepsilon_j} \kappa(X_{\varepsilon_1-\varepsilon_j}, X_{-(\varepsilon_1-\varepsilon_j)}) \\
&\quad + \frac{2c_o}{i+1} \sum_{j=i+1}^{n} N_{\varepsilon_1-\varepsilon_2,\varepsilon_2+\varepsilon_j} N_{\varepsilon_1-\varepsilon_j,-2\varepsilon_1} \eta_{\varepsilon_1-\varepsilon_j} \eta_{\varepsilon_2+\varepsilon_j} \kappa(X_{\varepsilon_1+\varepsilon_j}, X_{-(\varepsilon_1+\varepsilon_j)}) \\
&= \frac{2c_o}{i+1} \sum_{j=i+1}^{n} (1)(1) \eta_{\varepsilon_1+\varepsilon_j} \eta_{\varepsilon_2-\varepsilon_j} \kappa(X_{\varepsilon_1-\varepsilon_j}, X_{-(\varepsilon_1-\varepsilon_j)}) \\
&\quad + \frac{2c_o}{i+1} \sum_{j=i+1}^{n} (1)(-1) \eta_{\varepsilon_1-\varepsilon_j} \eta_{\varepsilon_2+\varepsilon_j} \kappa(X_{\varepsilon_1+\varepsilon_j}, X_{-(\varepsilon_1+\varepsilon_j)}) \\
&= \frac{4c_o^2}{i+1} \sum_{j=i+1}^{n} \eta_{\varepsilon_1+\varepsilon_j} \eta_{\varepsilon_2-\varepsilon_j} - \eta_{\varepsilon_1-\varepsilon_j} \eta_{\varepsilon_2+\varepsilon_j} \\
&= \frac{4c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2+\varepsilon_j}).
\end{aligned}$$

By a similar computation one obtains

$$\begin{aligned}
T_2 &= \frac{-2c_o}{i+1} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \kappa(X_{-(\varepsilon_1-\varepsilon_2)}, \mathrm{ad}(X)^2 X^*_{\gamma_j}) \kappa(X_{-2\varepsilon_2}, X_{\gamma_j}) \\
&= \frac{4c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2+\varepsilon_j}),
\end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{c_o}{i+1} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \sum_{k=3}^{i} \kappa(X_{-(\varepsilon_2-\varepsilon_k)}, \mathrm{ad}(X)^2 X^*_{\gamma_j}) \kappa(X_{-(\varepsilon_1+\varepsilon_k)}, X_{\gamma_j}) \\
&= \frac{2(i-2)c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2+\varepsilon_j}),
\end{aligned}$$

$$\begin{aligned}
T_4 &= \frac{-c_o}{i+1} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \sum_{k=3}^{i} \kappa(X_{-(\varepsilon_1-\varepsilon_k)}, \mathrm{ad}(X)^2 X^*_{\gamma_j}) \kappa(X_{-(\varepsilon_2+\varepsilon_k)}, X_{\gamma_j}) \\
&= \frac{2(i-2)c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j}) \kappa(X, X^*_{\varepsilon_2+\varepsilon_j}),
\end{aligned}$$

and

$$\begin{aligned} T_5 =& \frac{-c_o}{i+1} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \kappa(H_{\varepsilon_1-\varepsilon_2}, \mathrm{ad}(X)^2 X^*_{\gamma_j}) \kappa(X_{-(\varepsilon_1+\varepsilon_2)}, X_{\gamma_j}) \\ =& \frac{4c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j})\kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j})\kappa(X, X^*_{\varepsilon_2+\varepsilon_j}). \end{aligned}$$

Therefore $\tilde{\tau}_2(Y_l^*)(X)$ may be given as

$$\begin{aligned} &\tilde{\tau}_2(Y_l^*)(X) \\ &= T_1 + T_2 + T_3 + T_4 + T_5 \\ &= \frac{4c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j})\kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j})\kappa(X, X^*_{\varepsilon_2+\varepsilon_j}) \\ &\quad + \frac{4c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j})\kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j})\kappa(X, X^*_{\varepsilon_2+\varepsilon_j}) \\ &\quad + \frac{2(i-2)c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j})\kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j})\kappa(X, X^*_{\varepsilon_2+\varepsilon_j}) \\ &\quad + \frac{2(i-2)c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j})\kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j})\kappa(X, X^*_{\varepsilon_2+\varepsilon_j}) \\ &\quad + \frac{4c_o^2}{i+1} \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j})\kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j})\kappa(X, X^*_{\varepsilon_2+\varepsilon_j}) \\ &= 4c_o^2 \sum_{j=i+1}^{n} \kappa(X, X^*_{\varepsilon_1+\varepsilon_j})\kappa(X, X^*_{\varepsilon_2-\varepsilon_j}) - \kappa(X, X^*_{\varepsilon_1-\varepsilon_j})\kappa(X, X^*_{\varepsilon_2+\varepsilon_j}). \end{aligned} \tag{4.22}$$

Hence $\tilde{\tau}_2(Y_l^*)(X)$ is a non-zero polynomial on $\mathfrak{g}(1)$. □

4.3. **The special value.** Now we are going to find the special value for the $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system. As for the type 1b case, to find the special value, we use Proposition 2.23. Recall from Section 2.3 the linear map $\omega_2|_{V(\mu+\epsilon_\gamma)^*} : V(\mu+\epsilon_\gamma)^* \to \mathcal{U}(\bar{\mathfrak{n}})$ defined by $\omega_2|_{V(\mu+\epsilon_{n\gamma})^*} = \sigma \circ \tilde{\tau}_2|_{V(\mu+\epsilon_{n\gamma})^*}$, where $\sigma : \mathrm{Sym}^2(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ is the symmetrization operator. If Y_l^* is the lowest weight vector for $V(\mu+\epsilon_\gamma)^*$ defined in (4.14) then it follows from (4.22) that $\omega(Y_l^*) := \omega_{V(\mu+\epsilon_\gamma)^*}$ is given by

$$\omega(Y_l^*) = 4c_o^2 \sum_{j=i+1}^{n} \sigma(X^*_{\varepsilon_1+\varepsilon_j} X^*_{\varepsilon_2-\varepsilon_j}) - \sigma(X^*_{\varepsilon_1-\varepsilon_j} X^*_{\varepsilon_2+\varepsilon_j}).$$

By (4.2), this amounts to

$$\omega(Y_l^*) = \sum_{j=i+1}^{n} \sigma\big(X_{-(\varepsilon_1+\varepsilon_j)} X_{-(\varepsilon_2-\varepsilon_j)}\big) - \sigma\big(X_{-(\varepsilon_1-\varepsilon_j)} X_{-(\varepsilon_2+\varepsilon_j)}\big).$$

The following lemma will simplify arguments for a proof for Theorem 4.24 below.

Lemma 4.23. *For $X, Y, Z \in \mathfrak{g}$, in $\mathcal{U}(\mathfrak{g})$, we have*

$$X \cdot \sigma(YZ) = \sigma([X,Y]Z) + \sigma(Y[X,Z]).$$

Proof. A direct computation. □

Now we are ready to determine the special value for the $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system.

Theorem 4.24. *Let $\mathfrak{q}$ be the maximal parabolic subalgebra of type $C_n(i)$ for $2 \leq i \leq n-1$. The $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if $s = n-i+1$.*

Proof. By Proposition 2.23, to prove this theorem, it suffices to show that $X_\mu \cdot (\omega(Y_l^*) \otimes 1_{-s}) = 0$ in $M_{\mathfrak{q}}(\mathbb{C}_{-s})$ if and only if $s = n-i+1$, where μ is the highest weight for $\mathfrak{g}(1)$. It follows from (4.3) that $X_\mu \cdot (\omega(Y_l^*) \otimes 1_{-s})$ may be a sum of two terms. As $\mu = \varepsilon_1 + \varepsilon_{i+1}$, the first term is

$$T_1 = \sum_{j=i+1}^{n} \big(X_{\varepsilon_1+\varepsilon_{i+1}} \cdot \sigma(X_{-(\varepsilon_1+\varepsilon_j)}X_{-(\varepsilon_2-\varepsilon_j)})\big) \otimes 1_{-s}.$$

By Lemma 4.23, this may be expressed as

$$\begin{aligned} T_1 &= \sum_{j=i+1}^{n} \sigma([X_{\varepsilon_1+\varepsilon_{i+1}}, X_{-(\varepsilon_1+\varepsilon_j)}]X_{-(\varepsilon_2-\varepsilon_j)}) \otimes 1_{-s} \\ &\quad + \sum_{j=i+1}^{n} \sigma(X_{-(\varepsilon_1+\varepsilon_j)}[X_{\varepsilon_1+\varepsilon_{i+1}}, X_{-(\varepsilon_2-\varepsilon_j)}]) \otimes 1_{-s} \\ &= \sum_{j=i+1}^{n} \sigma([X_{\varepsilon_1+\varepsilon_{i+1}}, X_{-(\varepsilon_1+\varepsilon_j)}]X_{-(\varepsilon_2-\varepsilon_j)}) \\ &= \sigma(H_{\varepsilon_1+\varepsilon_{i+1}}X_{-(\varepsilon_2-\varepsilon_j)}) \otimes 1_{-s} \\ &\quad + \sum_{j=i+2}^{n} N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_1+\varepsilon_j)}\sigma(X_{-(\varepsilon_j-\varepsilon_{i+1})}X_{-(\varepsilon_2-\varepsilon_j)}) \otimes 1_{-s}. \end{aligned} \tag{4.25}$$

Since $\sigma(ab) = (1/2)(ab+ba)$, we have

$$\begin{aligned} &(4.25) \\ &= \sigma(H_{\varepsilon_1+\varepsilon_{i+1}}X_{-(\varepsilon_2-\varepsilon_j)}) \otimes 1_{-s} + \sum_{j=i+2}^{n} N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_1+\varepsilon_j)}\sigma(X_{-(\varepsilon_j-\varepsilon_{i+1})}X_{-(\varepsilon_2-\varepsilon_j)}) \otimes 1_{-s} \\ &= \frac{1}{2}\big(H_{\varepsilon_1+\varepsilon_{i+1}}X_{-(\varepsilon_2-\varepsilon_j)} + X_{-(\varepsilon_2-\varepsilon_j)}H_{\varepsilon_1+\varepsilon_{i+1}}\big) \otimes 1_{-s} \\ &\quad + \frac{1}{2}\sum_{j=i+2}^{n} N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_1+\varepsilon_j)}\big(X_{-(\varepsilon_j-\varepsilon_{i+1})}X_{-(\varepsilon_2-\varepsilon_j)} + X_{-(\varepsilon_2-\varepsilon_j)}X_{-(\varepsilon_j-\varepsilon_{i+1})}\big) \otimes 1_{-s}. \end{aligned} \tag{4.26}$$

If $\lambda_i = \sum_{j=1}^{i} \varepsilon_j$ is the fundamental weight for α_i then, as $H \cdot 1_{-s} = \lambda_i(H)1_{-s}$ for $H \in \mathfrak{h}$, a direct computation shows that

$$
\begin{aligned}
&(4.26)\\
&= \frac{1}{2}\big(H_{\varepsilon_1+\varepsilon_{i+1}} X_{-(\varepsilon_2-\varepsilon_j)} + X_{-(\varepsilon_2-\varepsilon_j)} H_{\varepsilon_1+\varepsilon_{i+1}}\big) \otimes 1_{-s}\\
&\quad + \frac{1}{2}\sum_{j=i+2}^{n} N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_1+\varepsilon_j)}\big(X_{-(\varepsilon_j-\varepsilon_{i+1})}X_{-(\varepsilon_2-\varepsilon_j)} + X_{-(\varepsilon_2-\varepsilon_j)}X_{-(\varepsilon_j-\varepsilon_{i+1})}\big) \otimes 1_{-s}\\
&= -\frac{1}{2}(\varepsilon_2-\varepsilon_{i+1})(H_{\varepsilon_1+\varepsilon_{i+1}})X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s} - s\lambda_i(H_{\varepsilon_1+\varepsilon_{i+1}})X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&\quad + \frac{1}{2}\sum_{j=i+2}^{n} N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_1+\varepsilon_j)} N_{\varepsilon_{i+1}-\varepsilon_j,-(\varepsilon_2-\varepsilon_j)} X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&= -\frac{1}{2}\ (1)\ X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s} - s\ (1)\ X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&\quad + \frac{1}{2}\sum_{j=i+2}^{n} (1)\ (1)\ X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&= -\frac{1}{2}(2s-n+i)X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}.
\end{aligned}
$$

Similarly, by a direct computation, the second term amounts to

$$
\begin{aligned}
T_2 &= -\sum_{j=i+1}^{n} \big(X_\mu \cdot \sigma(X_{-(\varepsilon_1-\varepsilon_j)}X_{-(\varepsilon_2+\varepsilon_j)})\big) \otimes 1_{-s}\\
&= -\frac{1}{2} N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_2+\varepsilon_{i+1})} N_{\varepsilon_1-\varepsilon_2,-(\varepsilon_1-\varepsilon_{i+1})} X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&\quad - \frac{1}{2}\sum_{j=i+1}^{n} N_{\varepsilon_1+\varepsilon_{i+1},-(\varepsilon_i-\varepsilon_j)} N_{\varepsilon_{i+1}+\varepsilon_j,-(\varepsilon_2+\varepsilon_j)} X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&= -\frac{1}{2}\ (1)\ (-1)\ X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&\quad - \frac{1}{2}\ (-2)\ (1)\ X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s} - \frac{1}{2}\sum_{j=i+2}^{n} (-1)\ (1)\ X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&= \frac{1}{2}(n-i+2)X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}.
\end{aligned}
$$

Therefore, $X_\mu \cdot (\omega(Y_l^*) \otimes 1_{-s})$ is given by

$$
\begin{aligned}
&X_\mu \cdot (\omega(Y_l^*) \otimes 1_{-s})\\
&= T_1 + T_2\\
&= -\frac{1}{2}(2s-n+i)X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s} + \frac{1}{2}(n-i+2)X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&= -\frac{1}{2}(2s-n+i-(n-i+2))X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}\\
&= -(s-(n-i+1))X_{-(\varepsilon_2-\varepsilon_{i+1})} \otimes 1_{-s}.
\end{aligned}
$$

Hence $X_\mu \cdot (\omega(Y_l^*) \otimes 1_{-s}) = 0$ if and only if $s = n-i+1$. $\square$

4.4. **The standardness of the map φ_{Ω_2}.** In the remainder of this section we determine the standardness of the map φ_{Ω_2} coming from the conformally invariant $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system.

Observe that if w_0 is the longest Weyl group element for $\mathfrak{l}_\gamma$ then the highest weight ν for $V(\mu+\epsilon_\gamma)^* = V(\varepsilon_1+\varepsilon_2)^*$ is $\nu = -w_0(\varepsilon_1+\varepsilon_2) = -\varepsilon_{i-1}-\varepsilon_i$. By Theorem 4.24, the special value s_2 for the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system is $s_2 = n-i+1$. Therefore, by (2.10), the $\Omega_2|_{V(\mu+\epsilon_{n\gamma})^*}$ system yields a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism

$$\varphi_{\Omega_2} : M_{\mathfrak{q}}((-\varepsilon_{i-1}-\varepsilon_i)-(n-i+1)\lambda_i+\rho) \to M_{\mathfrak{q}}(-(n-i+1)\lambda_i+\rho). \tag{4.27}$$

Theorem 4.28. *If $\mathfrak{q}$ is the maximal parabolic subalgebra of type $C_n(i)$ for $2 \le i \le n-1$ then the standard map φ_{std} between the generalized Verma modules in (4.27) is zero. Consequently, the map φ_{Ω_2} is non-standard.*

Proof. The idea of the proof is the same as for Theorem 3.14. Namely, first show that there exists $\alpha_\nu \in \Pi(\mathfrak{l})$ so that $-\alpha_\nu-(n-i+1)\lambda_i+\rho$ is linked to $(-\varepsilon_{i-1}-\varepsilon_i)-(n-i+1)\lambda_i+\rho$ and then apply Proposition 2.12. Observe that we have

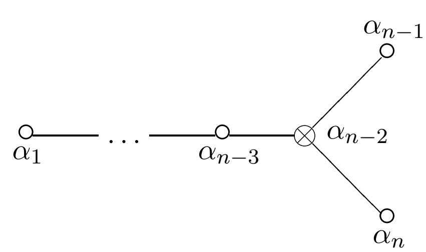
$$-\varepsilon_{i-1}-\varepsilon_i = -2(\varepsilon_i-\varepsilon_n)-(\varepsilon_{i-1}-\varepsilon_i)-2\varepsilon_n$$

with $\varepsilon_i-\varepsilon_n \in \Delta(\mathfrak{g}(1))$ and $\varepsilon_{i-1}-\varepsilon_i,\ 2\varepsilon_n \in \Pi(\mathfrak{l})$. (See Appendix B.) By a direct computation one can show that $(\varepsilon_{i-1}-\varepsilon_i, \varepsilon_i-\varepsilon_n)$ links $-2\varepsilon_n-(n-i+1)\lambda_i+\rho$ to $(-\varepsilon_{i=1}-\varepsilon_i)-(n-i+1)\lambda_i+\rho$. Now the theorem follows from Proposition 2.12. □

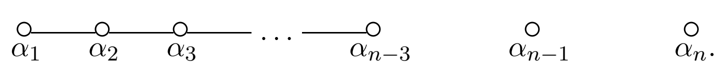
Appendix A. Parabolic subalgebra of type $D_n(n-2)$

In this appendix we observe the Ω_2 systems of maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ of type $D_n(n-2)$. In particular we find the special values for the Ω_2 systems and determine the standardness for the maps φ_{Ω_2}.

The parabolic subalgebra $\mathfrak{q}$ of type $D_n(n-2)$ is the maximal parabolic subalgebra determined by the simple root α_{n-2}; the deleted Dynkin diagram is



with subgraphs



As the simple root $\alpha_{\mathfrak{q}}$ that determines the parabolic subalgebra $\mathfrak{q}$ is $\alpha_{\mathfrak{q}} = \alpha_{n-2}$, the fundamental weight $\lambda_{\mathfrak{q}}$ for $\alpha_{\mathfrak{q}}$ is $\lambda_{\mathfrak{q}} = \lambda_{n-2}$.

Recall from Section 2.4 that the Levi subalgebra $\mathfrak{l}$ may be decomposed as

$$\mathfrak{l} = \mathbb{C}H_{\mathfrak{q}} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}^+_{n\gamma} \oplus \mathfrak{l}^-_{n\gamma}. \tag{A.1}$$

The unique simple root α_γ that is not orthogonal to the highest root γ is $\alpha_\gamma = \alpha_2$. Therefore we have $\mathfrak{l}_\gamma \cong \mathfrak{sl}(n-2,\mathbb{C})$ and $\mathfrak{l}_{n\gamma}^{\pm} \cong \mathfrak{sl}(2,\mathbb{C})$. For convenience we set $\mathfrak{l}_{n\gamma}^{+}$ (resp. $\mathfrak{l}_{n\gamma}^{-}$) to be the simple ideal that corresponds to the singleton for α_n (resp. α_{n-1}).

A.1. **Special constituents and special values.** We start with finding the special constituents of $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$. As shown in Section 2.4, the tensor product $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$ may be decomposed into

$$\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n}) = \big(\mathbb{C}H_{\mathfrak{q}}\otimes\mathfrak{z}(\mathfrak{n})\big)\oplus\big(\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})\big)\oplus\big(\mathfrak{l}_{n\gamma}^{+}\otimes\mathfrak{z}(\mathfrak{n})\big)\oplus\big(\mathfrak{l}_{n\gamma}^{-}\otimes\mathfrak{z}(\mathfrak{n})\big).$$

With the arguments in Section 6.1 of [14], one can easily check that $\mathbb{C}H_{\mathfrak{q}}\otimes\mathfrak{z}(\mathfrak{n})$ and $\mathfrak{l}_{n\gamma}^{\pm}\otimes\mathfrak{z}(\mathfrak{n})$ are simple $\mathfrak{l}$-modules. In fact, if we use the standard realization of roots with $\alpha_{n-1} = \varepsilon_{n-1}-\varepsilon_n$ and $\alpha_n = \varepsilon_{n-1}+\varepsilon_n$ then

$$\begin{aligned}
&\mathbb{C}H_{\mathfrak{q}}\otimes\mathfrak{z}(\mathfrak{n}) = V(\gamma) = V(\varepsilon_1+\varepsilon_2),\\
&\mathfrak{l}_{n\gamma}^{+}\otimes\mathfrak{z}(\mathfrak{n}) = V(\alpha_n+\gamma) = V(\varepsilon_1+\varepsilon_2+\varepsilon_{n-1}+\varepsilon_n), \text{ and}\\
&\mathfrak{l}_{n\gamma}^{-}\otimes\mathfrak{z}(\mathfrak{n}) = V(\alpha_{n-1}+\gamma) = V(\varepsilon_1+\varepsilon_2+\varepsilon_{n-1}-\varepsilon_n).
\end{aligned}$$

On the other hand, the tensor product $\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})$ is reducible. By using the character formula of Klimyk ([11, Corollary]), it can be shown that

$$\mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n}) = V(2\varepsilon_1+\varepsilon_2-\varepsilon_{n-2})\oplus V(\varepsilon_1+\varepsilon_2)\oplus V(2\varepsilon_1).$$

Now one may observe that only $V(2\varepsilon_1)$ and $V(\varepsilon_1+\varepsilon_2+\varepsilon_{n-1}\pm\varepsilon_n)$ satisfy the conditions (C1) and (C2) in Definition 2.21. Thus these irreducible constituents are the special constituents of $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$. Write ϵ_γ and $\epsilon_{n\gamma}^{\pm}$ for the roots in $\Delta(\mathfrak{g}(1))$ so that $\mu+\epsilon_\gamma = 2\varepsilon_1$ and $\mu+\epsilon_{n\gamma}^{\pm} = \varepsilon_1+\varepsilon_2+\varepsilon_{n-1}\pm\varepsilon_n$, where μ is the highest weight for $\mathfrak{g}(1)$. Tables 6 and 7 summarize the data for the special constituents.

TABLE 6. Roots μ, ϵ_γ, $\epsilon_{n\gamma}^{+}$, and $\epsilon_{n\gamma}^{-}$

Parabolic $\mathfrak{q}$	μ	ϵ_γ	$\epsilon_{n\gamma}^{+}$	$\epsilon_{n\gamma}^{-}$
$D_n(n-2)$	$\varepsilon_1+\varepsilon_{n-1}$	$\varepsilon_1-\varepsilon_{n-1}$	$\varepsilon_2+\varepsilon_n$	$\varepsilon_2-\varepsilon_n$

TABLE 7. Highest weights for special constituents

Parabolic $\mathfrak{q}$	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma}^{+})$	$V(\mu+\epsilon_{n\gamma}^{-})$
$D_n(n-2)$	$2\varepsilon_1$	$\varepsilon_1+\varepsilon_2+\varepsilon_{n-1}+\varepsilon_n$	$\varepsilon_1+\varepsilon_2+\varepsilon_{n-1}-\varepsilon_n$

Observe that μ, ϵ_γ, and $\epsilon_{n\gamma}^{\pm}$ are all long roots and that neither $\mu+\epsilon_\gamma$ nor $\mu+\epsilon_{n\gamma}^{\pm}$ is a root. Thus the special constituents $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma}^{\pm})$ are all of type 1a. (See Definition 2.22.) As $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$ contains a special constituent of type 1a, it follows from the argument in the proof for Proposition 7.3 of [14] that the τ_2 map is not identically zero. Also, the argument for Proposition 7.5 shows that, for $V(\mu+\epsilon) = V(\mu+\epsilon_\gamma), V(\mu+\epsilon_{n\gamma}^{\pm})$, the linear map $\tilde{\tau}_2|_{V(\mu+\epsilon)^*}$ is not identically zero.

Now we are going to find the special values of the Ω_2 systems arising from the special constituents $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon^{\pm}_{n\gamma})$. Recall from (3.4) that, for $\nu \in \mathfrak{h}^*$, we write

$$\Delta_\nu(\mathfrak{g}(1)) = \{\alpha \in \Delta(\mathfrak{g}(1)) \mid \nu - \alpha \in \Delta\}.$$

Theorem A.2. *Let $\mathfrak{q}$ be the maximal parabolic subalgebra of type $D_n(n-2)$. If $V(\mu+\epsilon) = V(\mu+\epsilon_\gamma)$ or $V(\mu+\epsilon^{\pm}_{n\gamma})$ then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if*

$$s = \frac{|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|}{2} - 1,$$

where $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|$ is the number of elements in $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))$.

Proof. Since the special constituents $V(\mu+\epsilon)$ and $V(\mu+\epsilon^{\pm}_{\gamma})$ are of type 1a, all the statements in [14] for type 1a special constituents can be applied. Now the theorem follows from the arguments in Section 7 in [14]. □

Corollary A.3. *Under the same hypotheses in Theorem A.2, all $\Omega_2|_{V(\mu+\epsilon)^*}$ systems are conformally invariant on $\mathcal{L}_1$.*

Proof. By inspection we have $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))| = 4$ for each special constituent. Now the results follow from Theorem A.2. □

A.2. **The standardness of the map φ_{Ω_2}.** In the rest of this section we determine whether or not the maps φ_{Ω_2} coming from the Ω_2 systems are standard.

Observe that

$$V(\mu+\epsilon_\gamma)^* = V(2\varepsilon_1)^* = V(-2\varepsilon_{n-2})$$

and

$$V(\mu+\epsilon^{\pm}_{n\gamma})^* = V(\varepsilon_1+\varepsilon_2+\varepsilon_{n-1}\pm\varepsilon_n)^* = V(-\varepsilon_{n-3}-\varepsilon_{n-2}+\varepsilon_{n-1}\pm\varepsilon_n).$$

It follows from Corollary A.3 that the special value s_2 is $s_2 = 1$ for each special constituent. Therefore if we denote by $\varphi_{(\Omega_2,\mu+\epsilon_\gamma)}$ (resp. $\varphi_{(\Omega_2,\mu+\epsilon^{\pm}_{n\gamma})}$) the homomorphism induced by the $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system (reps. the $\Omega_2|_{V(\mu+\epsilon^{\pm}_{n\gamma})^*}$ system) then, by (2.10), we have

$$\varphi_{(\Omega_2,\mu+\epsilon_\gamma)} : M_{\mathfrak{q}}(-2\varepsilon_{n-2}-\lambda_{n-2}+\rho) \to M_{\mathfrak{q}}(-\lambda_{n-2}+\rho) \tag{A.4}$$

and

$$\varphi_{(\Omega_2,\mu+\epsilon^{\pm}_{n\gamma})} : M_{\mathfrak{q}}((-\varepsilon_{n-3}-\varepsilon_{n-2}+\varepsilon_{n-1}\pm\varepsilon_n)-\lambda_{n-2}+\rho) \to M_{\mathfrak{q}}(-\lambda_{n-2}+\rho). \tag{A.5}$$

Theorem A.6. *If $\mathfrak{q}$ is the maximal parabolic subalgebra of type $D_n(n-2)$ then the standard maps between the generalized Verma modules in (A.4) and (A.5) are zero. Consequently, the maps $\varphi_{(\Omega_2,\mu+\epsilon_\gamma)}$ and $\varphi_{(\Omega_2,\mu+\epsilon^{\pm}_{n\gamma})}$ are non-standard.*

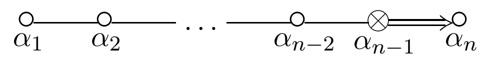
Proof. One can easily show that $(\varepsilon_{n-1}-\varepsilon_n, \varepsilon_{n-2}-\varepsilon_{n-1})$ links $-(\varepsilon_{n-1}+\varepsilon_n)-\lambda_{n-2}+\rho$ to $-2\varepsilon_{n-2}-\lambda_{n-2}+\rho$ and that $(\varepsilon_{n-3}-\varepsilon_{n-2}, \varepsilon_{n-2}-\varepsilon_{n-1})$ links to $(-\varepsilon_{n-3}-\varepsilon_{n-2}+\varepsilon_{n-1}\pm\varepsilon_n)-\lambda_{n-2}+\rho$. Now the theorem follows from Proposition 2.12. □

Appendix B. Miscellaneous Data

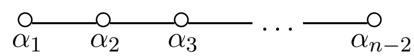
This appendix summarizes the miscellaneous data for the maximal parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ of types $B_n(n-1)$, $C_n(i)$ $(2 \leq i \leq n-1)$, and $D_n(n-2)$. For the data for other maximal parabolic subalgebras of quasi-Heisenberg type see for example Appendix A of [14].

§$\mathrm{B}_n(n-1)$

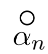
(1) The deleted Dynkin diagram:



(2) The subgraph for $\mathfrak{l}_\gamma$:



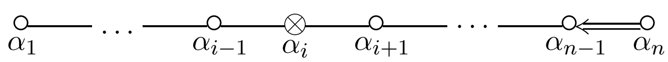
(3) The subgraph for $\mathfrak{l}_{n\gamma}$:



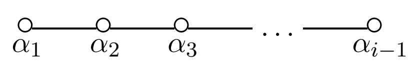
We have $\alpha_\gamma = \alpha_2$. The highest weight μ and the set of weights $\Delta(\mathfrak{g}(1))$ for $\mathfrak{g}(1)$ are $\mu = \varepsilon_1 + \varepsilon_n$ and $\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_n \mid 1 \leq j \leq n-1\} \cup \{\varepsilon_j \mid 1 \leq j \leq n-1\}$. The highest weight γ and the set of weights $\mathfrak{g}(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n}))$ are $\gamma = \varepsilon_1 + \varepsilon_2$ and $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \leq j < k \leq n-1\}$. The highest root ξ_γ and the set of positive roots $\Delta^+(\mathfrak{l}_\gamma)$ for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_{n-1}$ and $\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq n-1\}$. The highest root $\xi_{n\gamma}$ and the set of positive roots $\Delta^+(\mathfrak{l}_{n\gamma})$ for $\mathfrak{l}_{n\gamma}$ are $\xi_{n\gamma} = \varepsilon_n$ and $\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_n\}$.

§$\mathrm{C}_n(i)$, $2 \leq i \leq n-1$

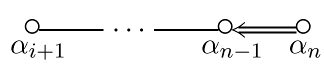
(1) The deleted Dynkin diagram:



(2) The subgraph for $\mathfrak{l}_\gamma$:



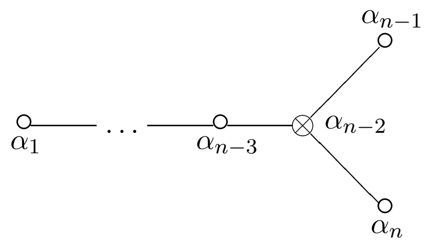
(3) The subgraph for $\mathfrak{l}_{n\gamma}$:



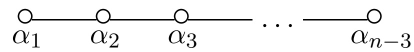
We have $\alpha_\gamma = \alpha_1$. The highest weight μ and the set of weights $\Delta(\mathfrak{g}(1))$ for $\mathfrak{g}(1)$ are $\mu = \varepsilon_1 + \varepsilon_{i+1}$ and $\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\}$. The highest weight γ and the set of weights $\Delta(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n})$ are $\gamma = 2\varepsilon_1$ $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \leq j < k \leq i\} \cup \{2\varepsilon_j \mid 1 \leq j \leq i\}$. The highest root ξ_γ and the set of positive roots $\Delta^+(\mathfrak{l}_\gamma)$ for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_i$ and $\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq i\}$ The highest root $\xi_{n\gamma}$ and the set of positive roots $\Delta(\mathfrak{l}_{n\gamma})$ for $\mathfrak{l}_{n\gamma}$ are $\xi_{n\gamma} = 2\varepsilon_{i+1}$ and $\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_j \pm \varepsilon_k \mid i+1 \leq j < k \leq n\} \cup \{2\varepsilon_j \mid i+1 \leq j \leq n\}$.

§$\mathrm{D}_n(n-2)$

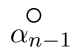
(1) The deleted Dynkin diagram:



(2) The subgraph for $\mathfrak{l}_\gamma$:



(3) The subgraph for $\mathfrak{l}_{n\gamma}^-$:



(4) The subgraph for $\mathfrak{l}_{n\gamma}^+$:

We have $\alpha_\gamma = \alpha_2$. The highest weight μ and the set of weights $\Delta(\mathfrak{g}(1))$ for $\mathfrak{g}(1)$ are $\mu = \varepsilon_1 + \varepsilon_{n-1}$ and $\Delta(\mathfrak{g}(1)) = \{e_j \pm e_k \mid 1 \leq j \leq n-2 \text{ and } k = n-1, n\}$. The highest weight γ and the set of weights $\Delta(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n})$ are $\gamma = \varepsilon_1 + \varepsilon_2$ and $\Delta(\mathfrak{z}(\mathfrak{n})) = \{e_j + e_k \mid 1 \leq j < k \leq n-2\}$. The highest root ξ_γ and the set of positive roots $\Delta^+(\mathfrak{l}_\gamma)$ for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_{n-2}$ and $\Delta^+(\mathfrak{l}_\gamma) = \{e_j - e_k \mid 1 \leq j < k \leq n-2\}$. The highest root $\xi_{n\gamma}^-$ and the set of positive roots $\Delta^+(\mathfrak{l}_{n\gamma}^-)$ are $\xi_{n\gamma}^- = \varepsilon_{n-1} - \varepsilon_n$ and $\Delta^+(\mathfrak{l}_{n\gamma}^-) = \{\varepsilon_{n-1} - \varepsilon_n\}$. The highest root $\xi_{n\gamma}^+$ and the set of positive roots $\Delta^+(\mathfrak{l}_{n\gamma}^+)$ are $\xi_{n\gamma}^+ = \varepsilon_{n-1} + \varepsilon_n$ and $\Delta^+(\mathfrak{l}_{n\gamma}^+) = \{\varepsilon_{n-1} + \varepsilon_n\}$.

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